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Diffraction of Scalar Waves by a Circular Aperture II

J. BAZER and H. HOCHSTADT

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DIFFRACTION OF SCALAR WAVES
BY A CIRCULAR APERTURE II

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ABSTRACT

The problems of diffraction of scalar waves by a circular aperture in a perfectly soft and in a perfectly rigid infinite planar screen are studied. Excitation of the form $\cos m(\phi-\phi_o^{(m)})u^{(m)}(r,z)$, $m = 0,1,2,\ldots,$ originating in the region $z < 0$ is assumed. Here, with the z-axis taken as the axis of symmetry, r, ϕ, and z denote the cylindrical coordinates of an arbitrary space point and $\phi_o^{(m)}$ is an arbitrary phase angle. On the shadow side of the screen, the solution $V_{1,2}$ is represented as follows:

$$V_{1,2} = \cos m(\phi-\phi_o^{(m)})r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \int_{-1}^{1} \left\{\exp\left[ikR^+(r,z;t)\right]/R^+(r,z;t)\right\} f_{1,2}^{(m)}(t)dt.$$

The subscripts 1 and 2 in this expression refer to the solutions for the soft and rigid screen problems respectively; k is the wave number

$$R^+(r,z;t) = \sqrt{r^2 + (z + iat)^2};$$

$z = 0$ defines the plane of the screen; 'a' is the radius of the aperture, centered at the origin. The unknown functions $f_1^{(m)}(t)$ and $f_2^{(m)}(t)$ are required, respectively, to be odd and even analytic functions of t, to be regular in a neighborhood of the unit disc, $0 \leq |t| \leq 1$, and to have zeros of order m, at $t = \pm 1$. These integral representations provide the natural generalizations of the corresponding axially symmetric representations ($m = 0$ in the above) that were presented in an earlier work. As in the axially-symmetric case, they are designed to satisfy the time-reduced wave equation, the boundary conditions, the radiation condition and to have the proper edge behavior, assuming merely that the f's exist. A straightforward calculation reveals the f's to be solutions of a Fredholm-type integral equation of the second kind. Whenever the excitation is sufficiently regular in a neighborhood of the aperture, the integral equations guarantee, for ka small, the existence of the desired types of f's and, moreover, furnish approximate (truncated Neumann series) expressions for them. Application of these results is made to the special case of plane-wave excitation. Approximate expressions, in powers of ka, are derived for the far fields, transmission coefficients, aperture fields and edge fields of the solutions associated with the first three modes ($m = 0.1,2$) in the Fourier-Bessel expansion of the plane wave; the leading term as well as several higher order terms are given for each quantity. Analogous expressions are derived for $m \geqq 3$; but here only the leading terms are given.

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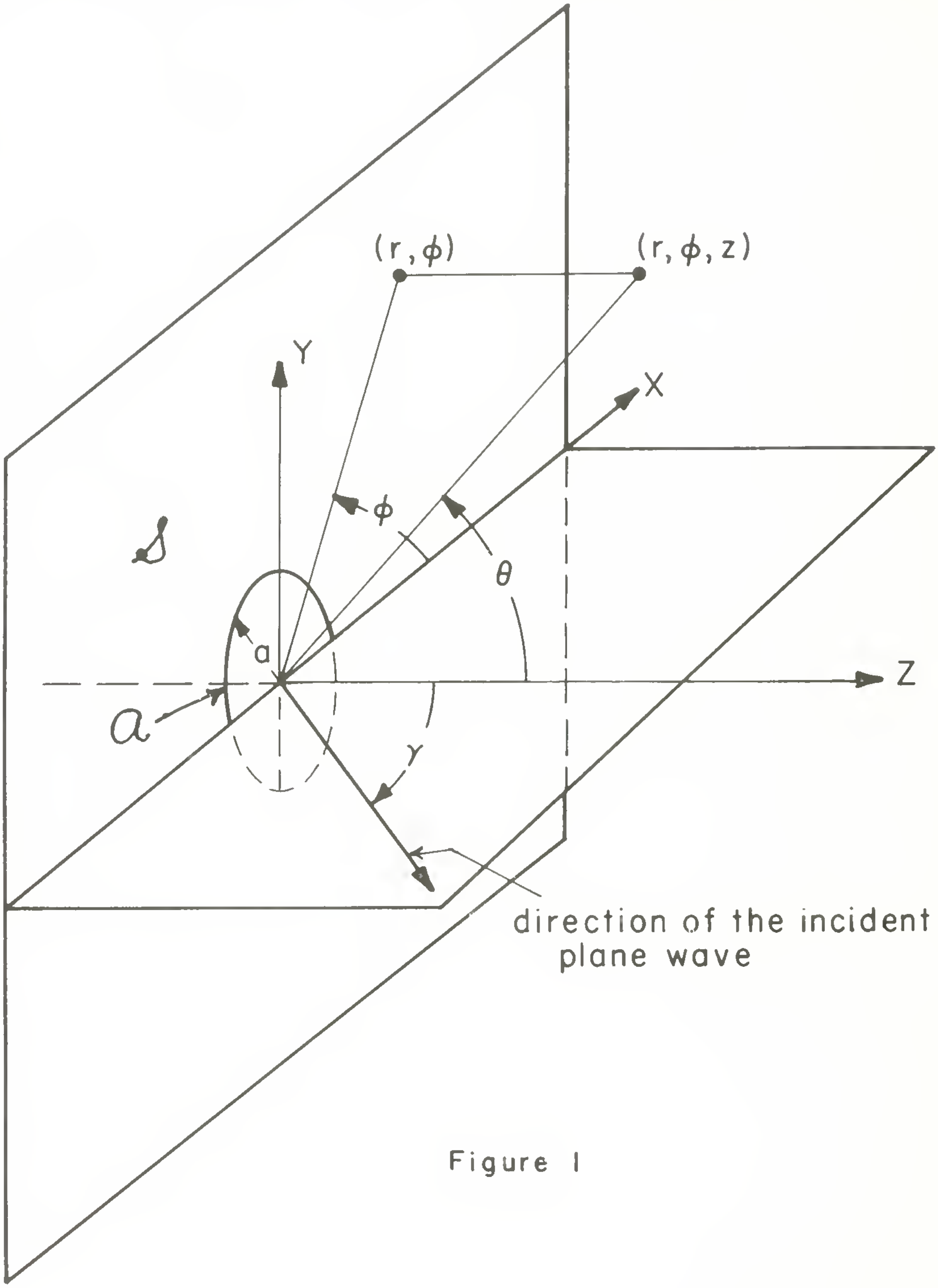

Figure 1

A view of the diffracting screen. $\mathcal{S}$ is the screen, $\mathcal{A}$ is the aperture of radius 'a' centered at the origin of coordinates. $\mathcal{S} + \mathcal{A}$ coincides with the (x,y)-plane. The symbols r, ϕ, and z are cylindrical coordinates of an arbitrary point of space; r denotes the distance from the z-axis (the axis of symmetry); ϕ is measured from the positive x-axis in the (x,y)-plane. The angle θ is measured from the positive z-axis to the ray that connects the origin to (r,ϕ,z). The angle γ fixes the direction of propagation of the incident plane wave.

1. INTRODUCTION

We are concerned with the problems of diffraction by a circular aperture in a perfectly soft and in a perfectly rigid infinite, planar screen. Our attention is focused primarily on the low frequency end of the spectrum where the product ka ($=\alpha$) of the wave number k and the aperture radius 'a' is small. The solutions of J. Bazer and A. Brown[1] for axially-symmetric excitation are herein generalized to cover the case of non-axially symmetric excitation consisting of a superposition of 'modes' of the form $\cos[m(\phi-\phi_0^{(m)})]u^{(m)}(r,z)$, $m = 0,1,2,\ldots$. In this expression, r, ϕ and z denote the cylindrical coordinates of an arbitrary space point, with the z-axis serving as the axis of symmetry [see Figure 1]; $\phi_0^{(m)}$ is an arbitrary phase angle. As in the axially-symmetric case, the new solutions take the form of integral representations, $\mathcal{Q}_1^{(m)}(f_1^{(m)})$ and $\mathcal{Q}_2^{(m)}(f_2^{(m)})$ say*, which are related to certain unknown functions $f_j^{(m)}(t)$, $j = 1,2$ by means of single quadratures. These representations are designed to satisfy automatically the time-reduced wave equation, the boundary conditions, the Sommerfeld radiation condition and to have the proper edge behavior. According as the first or second boundary-value problem is under consideration, the condition that the normal derivative of the solution or that the solution itself be continuous through the aperture then leads to Fredholm integral equations of the second kind for for $f_1^{(m)}$ and $f_2^{(m)}$. Whenever the excitation is sufficiently regular in the neighborhood of the aperture, these integral equations guarantee, for sufficiently small α, the existence of $f^{(m)}$'s of the desired type and, moreover, furnish approximate (truncated Neumann series) expressions for these functions. We apply these results to the special case of plane-wave excitation. Specifically, for $m = 0$, 1 and 2 we make use of the approximate $f^{(m)}$'s to obtain approximate expressions, in powers of α, for the aperture fields, far fields, edge fields and transmission coefficients; the leading terms as well as higher order terms in α are given. These expressions evidently derive from the solutions associated with the first three harmonics in the Fourier-Bessel expansion of the plane wave. Analogous expressions are derived for the higher harmonics, $m \geq 3$, in this expansion; but here only the leading term is given for each of the above-mentioned field characteristics. When these results are further specialized to the (axially-symmetric) case of normally-incident plane wave excitation, they are found to agree with known results* obtained by other means. Moreover, it appears that some of these results, chiefly those associated with higher harmonic excitation, are new.**

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*Here, and in what follows, it is assumed that the excitation consists of a single mode. The superscript m identifies which harmonic is under discussion. Quantities relating to the soft-screen problem or first boundary-value problem are distinguished from their analogs in the rigid-screen or second boundary-value problem by appending the subscripts 1 and 2.

The integral representations for the higher harmonics constitute the essentially new element of this paper from which all other results follow. As will be seen, these representations can be generated in a natural way from the axially-symmetric solutions presented in reference [1]. The axially-symmetric solutions are, in turn, the analogs of similarly represented solutions of the first and second boundary-value problems of potential theory. We refer the reader to the book by Green and

---

*See, for example, C.J. Bouwkamp's papers in reference [2].

**See the last paragraphs of Sections 5 and 7.

Zerna[3] and especially to the recent work of W.D. Collins[4a,b] where similar representations have been exploited to solve some interesting problems in potential theory. Several authors have succeeded in reducing aperture and disc problems with axial symmetry and α small to the problem of solving Fredholm integral equations of the second kind. Among these are W. Magnus[5], N. Chako[6], D.S. Jones[7], N.I. Akhiezer and A.N. Akhiezer[8], B. Noble[9], R.C. McCamy and A.E. Heins[10], and W.D. Collins[4c].* Our integral equations for the zeroth harmonic are most closely related to one first obtained by D.S. Jones[7].** The non-axially symmetric extensions of these solutions have not, to our knowledge, been published.*** Mention should here be made of J.B. Keller's geometrical theory[11] which requires, to be sure, that α be 'large', but is limited neither to axially-symmetric excitation nor to circular apertures. It turns out that the formulas for the aperture fields, far fields and transmission coefficients calculated in accordance with this theory yield accurate results for values of α as low as $\alpha = 1$. Most other attacks on aperture and disc problems have been carefully surveyed by C.J. Bouwkamp[2].

2. FORMULATION OF THE PROBLEM

Let $\mathcal{A}$ be a circular aperture of radius 'a' in a planar screen $\mathcal{S}$ and let $\mathcal{A} + \mathcal{S}$ coincide with the (x,y)-plane, the origin being located at the center of $\mathcal{A}$ [see Figure 1]. Then the solution U_1 of the first boundary-value problem may be expressed as follows [cf. reference [2b], p. 38]:

---

* This work deals with the problem of diffraction by coaxial parallel discs.

** Jones treats the rigid disc problem which, by virtue of Babinet's principle, is equivalent to the first boundary-value problem for the aperture.

*** In a private communication, W.D. Collins has informed us that he has independently arrived at our solution for the case m = 1 in a study dealing with the torsional oscillations set up in a half space (elastic) by a disc.

$$U_1 = U_1(r,\phi,z) = \begin{cases} U^o(r,\phi,z) - U^o(r,\phi,-z) + V_1(r,\phi,-z), & z \leqq 0 \\ V_1(r,\phi,z), & z \geqq 0. \end{cases} \tag{2.1}$$

In this equation, the triplet (r,ϕ,z) denotes the cylindrical coordinates of an arbitrary point in space. The angular coordinate ϕ is measured from the positive direction of the x-axis, as shown in Figure 1, so that

$$\begin{aligned} x &= r \cos \phi \;, \\ y &= r \sin \phi \;, & 0 \leqq \phi < 2\pi. \end{aligned} \tag{2.2}$$

The incident wave, that is, U_o, is assumed to impinge upon the screen from the left (in the half space $z < 0$). The function V_1 describes the scattered field; it is defined for all $z \geqq 0$ and is required to have the following properties:

(2.3)

(i) V_1 is a solution of the wave equation $\Delta V + k^2 V = 0$;

(ii) $V_1 = 0$ on $\mathcal{S}$;

(iii) V_1 satisfies the Sommerfeld radiation condition at infinity;

(iv) $\dfrac{\partial V_1}{\partial z} = \dfrac{\partial U^o}{\partial z}$ in $\mathcal{A}$;

(v) V_1 is everywhere finite;

(vi) ∇V_1 is quadratically integrable over any domain in 3-space, including domains having some or all points of the circular edge as boundary points.

Similarly, the solution $U_2(r,\phi,z)$ of the second boundary-value problem may be expressed as follows.

$$(2.4) \qquad U_2(r,\phi,z) = \begin{cases} U^o(r,\phi,z) + U^o(r,\phi,-z) - V_2(r,\phi,-z), & z \leqq 0 \\ V_2(r,\phi,z), & z \geqq 0. \end{cases}$$

Here, the scattered wave function V_2 is defined in the half-space $z \geqq 0$ and is required to have all the properties listed above except (ii) and (iv) which must be replaced by

$$(2.5) \qquad \begin{aligned} &(ii)' \quad \frac{\partial V_2}{\partial z} = 0 \text{ on } \mathcal{S}, \\ &(iv)' \quad V_2 = U^o \text{ in } \mathcal{A} . \end{aligned}$$

In both the first and second boundary-value problems, the incident wave will be assumed to have the form

$$(2.6) \qquad U^o(r,\phi,z) = \cos m(\phi-\phi_o{}^{(m)})u^{(m)}(r,z), \qquad m = 0,1,2,\ldots$$

where $\phi_o{}^{(m)}$ denotes an arbitrary phase constant and $u^{(m)}(r,z)$ is a function of r and z alone. More general excitation can be formed from a superposition of such 'modes'.*

---

*Questions relating to the convergence of this and the corresponding Fourier series for the total wave function U_j, $j = 1,2$, when an infinite number of modes are involved, will be left open.

For the purposes of the ensuing analysis, it is necessary to impose suitable conditions on the incident wave at $z = 0$. To formulate these conditions, we first write

$$u^{(m)}(r,0) = r^m u_2^{(m)}(r) \quad , \quad m = 0,1,2,\dots , \tag{2.7}$$

and

$$\left.\frac{\partial u^{(m)}(r,z)}{\partial z}\right|_{z\downarrow 0} = r^m u_1^{(m)}(r) \quad , \quad m = 0,1,2,\dots , \tag{2.8}$$

for the $u^{(m)}$ and their normal derivatives in the aperture. In terms of the $u_k^{(m)}$'s, $k = 1,2$, $m = 0,1,2,\dots$, the aforementioned conditions may now be expressed as follows: It is required (1) that the $u_k^{(m)}(r)$'s be even functions of r, (2) that they be analytic functions of r whose analytic continuations in the complex r-plane are regular at least in a (closed) circular domain of radius $a + \Delta$ $(\Delta > 0)$ centered at $r = 0$ and finally, (3) that they be absolutely integrable over every finite r-interval and $\mathcal{O}[e^{Ar}]$ $(A > 0)$ for large r.*

Consider, for example, the cases of plane-wave and spherical-wave excitation. In cylindrical coordinates one has the formulas**

plane wave:

$$U_0(r,\phi,z) = e^{ik[z\cos\gamma + r\sin\gamma\cos\phi]} \tag{2.9}$$

$$= e^{ikz\cos\gamma} \sum_{m=0}^{\infty} i^m \epsilon_m r^m \left(\frac{J_m(kr\sin\gamma)}{r^m}\right) \cos m\phi$$

---

*This condition is imposed so that the Laplace transforms of the $u_k^{(m)}(r)$ and of certain related functions may exist.

**Time dependence of the form $\exp(-i\omega t)$ is assumed throughout, ω denoting the angular frequency.

spherical wave:

$$(2.10)\quad U_o(r,\phi,z) = \frac{e^{ikR}}{R}$$

$$= \sum_{m=o}^{\infty} \epsilon_m \cos m(\phi-\phi_o) r^m \int_o^{\infty} \frac{J_m(\lambda r) J_m(\lambda r_o) e^{-|z-z_o|\sqrt{\lambda^2-k^2}}}{r^m \sqrt{\lambda^2-k^2}} \lambda d\lambda ,$$

where $z_o < 0$,

$$(2.11)\qquad \epsilon_m = \begin{cases} 1, & m = 0 \quad , \\ 2, & m \geq 1 \quad , \end{cases}$$

and

$$(2.12)\qquad R = \sqrt{r^2 + r_o^2 - 2rr_o \cos(\phi-\phi_o) + (z-z_o)^2} \ .$$

In the expressions for the plane wave, it has been assumed that the direction of incidence is in the (x,z)-plane and makes an angle γ, $0 \leq \gamma < \pi/2$, with the z-axis [see Figure 1]. In equation (2.10) the triplet (r_o, ϕ_o, z_o) denotes the coordinates of the point source.

Comparing the appropriate harmonics in equations (2.9) and (2.10) with the right member of equation (2.6) one finds, in view of equations (2.7) and (2.8), that

plane wave*:

$$(2.13)\qquad u_1^{(m)}(r) = ik\cos\gamma u_2^{(m)}(r) = i^{m+1} k\cos\gamma \epsilon_m J_m(kr\sin\gamma)/r^m =$$

---

*The last equation follows from the recursion formula $(-\frac{1}{z}\frac{d}{dz})^m Z_o(z) = Z_m(z)/z^m$, which holds for all cylindrical $Z_m(z)$ functions of integral order m.

$$= \frac{ik\cos\gamma\epsilon_m(-i)^m}{(k\sin\gamma)^m} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m J_o(kr\sin\gamma)$$

spherical wave:

$$(2.14) \qquad u_1^{(m)}(r) = -\epsilon_m \int_0^\infty \frac{J_m(\lambda r)J_m(\lambda r_o)e^{z_o\sqrt{\lambda^2-k^2}}}{r^m} \lambda d\lambda, \qquad z_o < 0 ;$$

$$(2.15) \qquad u_2^{(m)}(r) = \epsilon_m \int_0^\infty \frac{J_m(\lambda r)J_m(\lambda r_o)e^{z_o\sqrt{\lambda^2-k^2}}}{r^m\sqrt{\lambda^2-k^2}} \lambda d\lambda, \qquad z_o < 0 .$$

The $u_k^{(m)}(r)$'s in (2.13) clearly have all the properties listed above. The same can be shown to hold true for the $u_k^{(m)}$'s of equations (2.14) and (2.15). From this last remark it follows that our requirements on the behavior of the incident field entail no essential loss of generality; for any localized source may be regarded as a superposition of spherical waves.

Finally, it may be mentioned that the problems of diffraction by perfectly soft and perfectly rigid discs of radius 'a' can be solved in terms of the functions V_2 and V_1 introduced in equations (2.1) and (2.4). In fact, it can be shown that (see reference [2b],p.39) that

$$(2.16) \qquad U_{1,d} = \begin{cases} U_o(r,\phi,z) - V_2(r,\phi,-z) , & z < 0 , \\ U_o(r,\phi,z) - V_2(r,\phi,z) , & z > 0 , \end{cases}$$

and

$$(2.17) \qquad U_{2,d} = \begin{cases} U_o(r,\phi,z) + V_1(r,\phi,-z) \ , & z < 0 \\ U_o(r,\phi,z) - V_1(r,\phi,z) \ , & z > 0. \end{cases}$$

where $U_{j,d}$ denotes the solution of the soft or rigid-disc problem according as $j = 1$ or 2.

3. INTEGRAL REPRESENTATIONS OF THE SCATTERED WAVE V_1

In reference [1], where only the case of axially-symmetric excitation was studied, the scattered wave $V_1 = V_1^{(o)}$ was assumed to have the form

$$V_1^{(o)} = \int_{-1}^{1} \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} f_1^{(o)}(t)dt \tag{3.1}$$

$$= \int_{o}^{1} \left\{ \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} - \frac{e^{ikR^-(r,z;t)}}{R^-(r,z;t)} \right\} f_1^{(o)}(t)dt \ ,$$

where $R^+(r,z;t)$ and $R^-(r,z;t)$ were defined by

$$R^{\pm}(r,z;t) = \sqrt{r^2 + (z \pm iat)^2} \ , \tag{3.2}$$

the upper and lower signs on the left corresponding respectively to the upper and lower signs on the right. The function $f_1^{(o)}(t)$ was required to be (1) an odd function* of t and (2), a regular function of t in the circular domain defined by the relation $0 \leqq |t| \leqq 1 + \Delta/a$. Since the regularity of the incident wave U^o in the domain $0 \leqq |r| \leqq a + \Delta$, $\Delta > 0$, [cf. requirement (2) following equation (2.8) of the previous section] ultimately implied that $f_1^{(o)}(t)$ satisfied property (2), no generality was lost in requiring that $f_1^{(o)}(t)$ enjoy this property at the very outset.

We wish now to present a generalization of the above Ansatz. Owing to our assumption about the form of the incident wave [see equation (2.6)], we may reduce the problem to that of finding functions

---

*Use has already been made of this property in equation (3.1).

$$(3.3) \qquad V_1 = \cos m(\phi-\phi^{(m)})V_1^{(m)}(r,z) , \qquad m = 0,1,2,\ldots$$

which for each m satisfy conditions (i), (ii), (iii), (v), and (vi) of equation (2.3) and the relation

$$(3.4) \qquad \left.\frac{\partial V_1^{(m)}(r,z)}{\partial z}\right|_{z\downarrow 0} = r^m u_1^{(m)}(r) , \qquad 0 \leq r < a,$$

in the aperture $\mathcal{A}$. To find such functions we proceed as follows. Let V(x,y,z) be any solution of the wave equation

$$(3.5) \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0 , \qquad z > 0 .$$

Then, continuing in a purely formal manner, we find that the functions D_+V and D_-V defined by

$$(3.6) \qquad D_\pm V = \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y}\right)V$$

are likewise solutions. In fact, $(D_\pm)^m V$ defined by

$$(3.7) \qquad (D_\pm)^m V = \overbrace{\left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y}\right) \ldots\left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y}\right)}^{|\longleftarrow \text{ m factors } \longrightarrow|}V$$

are solutions of equation (3.7) for each m, m = 1,2,... . Moreover, if V vanishes on the screen, then $(D_\pm)^m V$ will also vanish on the screen. The application of the D_+'s and D_-'s on V may of course produce unwanted singularities in the resulting wave function at the circular edge, but we shall ignore this possibility for the moment.

Representing $D_\pm$ in polar coordinates [see equation (2.2)], we arrive at the operator identity

$$(3.8) \qquad D_\pm = e^{\pm i\phi}\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial\phi}\right) .$$

If now $(D_\pm)^m$ is applied to any axially-symmetric solution - $W = W(r,z)$ say - of the wave equation, it is found that

$$(3.9) \qquad (D_\pm)^m W = e^{\pm im\phi}\left[\frac{\partial}{\partial r} - \frac{(m-1)}{r}\right]\left[\frac{\partial}{\partial r} - \frac{(m-2)}{r}\right]\cdots\left[\frac{\partial}{\partial r} - \frac{1}{r}\right]W(r,z) ,$$

which may be written more compactly as

$$(3.10) \qquad (D_\pm)^m W = e^{\pm im\phi} r^m \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m W.$$

The above considerations and this last identity suggest that the proper representation for the scattered field V_1 is:

$$(3.11) \quad V_1 = \cos m(\phi-\phi_o^{(m)})V_1^{(m)} = \cos m(\phi-\phi_o^{(m)})\left[r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \int_{-1}^{1} \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} f_1^{(m)}(t)dt\right].$$

In effect, we have replaced W in equation (3.9) by an axially symmetric wave function having precisely the same form as the scattered wave $V_1^{(o)}$ of the axially-symmetric solution [cf. equation (3.1)].

The functions $f_1^{(m)}(t)$, $m = 1,2,\ldots$, are required to share with $f_1^{(o)}(t)$ the property of being (1) odd as functions of t and (2) regular functions of t in the circular domain defined by the relation $0 \leqq |t| \leqq 1 + \Delta/a$ [see the text following equation (3.2)]. In addition, to ensure the proper edge behavior, it is necessary to insist that for all $m \geqq 1$

$$\left. \frac{d^k f_1^{(m)}(t)}{dt^k} \right|_{t=1} = 0 , \qquad k = 0,1,\ldots,m-1 , \tag{3.12}$$

which, on account of the oddness of $f_1^{(m)}(t)$, implies that

$$\left. \frac{d^k f_1^{(m)}(t)}{dt^k} \right|_{t=-1} = 0 , \qquad k = 0,1,\ldots,m-1 . \tag{3.13}$$

In the above, as in what follows, it is to be understood that

$$\lim_{z \downarrow 0} R^{\pm}(r,z;t) = +\sqrt{r^2 - a^2t^2} , \text{ when } r^2 > a^2t^2. \tag{3.14}$$

This specification of the branches of $R^{\pm}(r,z;t)$ immediately implies that

$$\lim_{z \downarrow 0} R^{\pm}(r,z;t) = \pm\sqrt{a^2t^2 - r^2} , \text{ when } r^2 < a^2t^2 . \tag{3.15}$$

At points in the aperture, the only points in physical space where the integrand of (3.11) may be singular, $V_1^{(m)}$ and its first and higher order derivatives are defined to be the limits approached when z approaches zero through positive values. The limits themselves can be derived with the aid of the properties of $f_1^{(m)}(t)$ listed above and Cauchy's Theorem;

Appendix I of reference [1] relating to the calculation of the edge behavior, exemplifies the sort of argument required*. For the sake of brevity, we shall not justify the ensuing limiting processes. It is enough to say that the several limiting processes employed in what follows are justifiable and agree with those which are arrived at on the basis of purely formal considerations.

Since, in Sommerfeld's notation, we have the relations**

$$\zeta_o^{(1)}(w) = \sqrt{\frac{\pi}{2w}}\; H_{\frac{1}{2}}^{(1)}(w) = \frac{e^{iw}}{iw}, \tag{3.16}$$

$$\zeta_n^{(1)}(w) = \sqrt{\frac{\pi}{2w}}\; H_{n+\frac{1}{2}}^{(1)}(w), \tag{3.17}$$

$$\left(-\frac{1}{w}\frac{d}{dw}\right)^m \zeta_o^{(1)}(w) = \frac{\zeta_m^{(1)}(w)}{w^m}, \tag{3.18}$$

and since for each natural number m we have the identities

$$\begin{aligned}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \zeta_o^{(1)}(kR^+) &= \left(\frac{1}{R^+}\frac{d}{dR^+}\right)^m \zeta_o^{(1)}(kR^+)\\ &= k^{2m}\left[\frac{1}{kR^+}\frac{d}{d(kR^+)}\right]^m \zeta_o^{(1)}(kR^+),\end{aligned} \tag{3.19}$$

we may rewrite equation (3.11) as

---

* The requirement in property (2) that $f_1^{(m)}(t)$ be regular in the domain $0 \leq |t| \leq 1 + \Delta/a$ is for example, essential here.

** See equations (23)-(35), pp 404-406 in J.A. Stratton's book [12]. Note that Stratton's $h_n^{(1)}(w)$ corresponds to Sommerfeld's $\zeta_n^{(1)}(w)$.

$$(3.20) \qquad V_1 = \cos m(\phi-\phi_o{}^{(m)})V_1{}^{(m)}$$

$$= \cos m(\phi-\phi_o{}^{(m)})\left[ik^{2m+1}(-r)^m \int_{-1}^{1} \frac{\zeta_m{}^{(1)}[kR^+(r,z;t)]}{[kR^+(r,z;t)]^m} f_1{}^{(m)}(t)dt\right]$$

whenever the relations $z > 0$, or $z \geqq 0$ and $r > a$ hold.

If it is assumed merely that the $f_1{}^{(m)}$'s of the desired type exist, then it can be shown, even at this stage, that the corresponding functions, $V_1 = \cos m(\phi-\phi_o{}^{(m)})V_1{}^{(m)}$, $m = 0,1,\ldots,$ satisfy all the conditions of equation (2.3) with the exception of the aperture condition [see equation (3.4)]. That V_1 satisfies the wave equation (3.7) in the half-space $z > 0$ follows directly from the fact that $(D_\pm)^m\left\{\exp(ikR^+)/R^+\right\}$, and linear combinations of these functions enjoy this property for each $m \geqq 0$. To verify that V_1 vanishes on the screen [condition (ii)] we need only note that the integrand of equation (3.11) is non-singular when $r > a$ $(z \geqq 0)$ and that the $f_1{}^{(m)}$'s are odd functions of t; the vanishing of the V_1's then follows immediately on letting z approach zero. Next, we write

$$(3.21) \qquad \begin{aligned} r &= R\sin\theta \;, \\ z &= R\cos\theta \;, \qquad\qquad 0 \leqq \theta \leqq \pi/2 \;, \\ R &= r^2 + z^2 \;, \end{aligned}$$

where θ is the angle which the vector (x,y,z) makes with the positive z-axis [see Figure 1] and note that

$$(3.22) \qquad R^\pm(r,z;t) = R \pm iat\cos\theta + \mathcal{O}(R^{-1})$$

when R is large. The Sommerfeld radiation condition [condition (iii)] is now seen to follow from this relation and the known asymptotic properties of the $\zeta_m^{(1)}$-functions [see equations (3.19) and (3.20)] or directly from the asymptotic form of the last member of equation (3.11).

Since $\exp[ikR^+]/R^+$ and its r and z-derivatives are continuous functions of t in the half-space $z > 0$, we conclude from equation (3.11) that for each $m \geqq 0$ V_1 and grad V_1 are finite in that region - the singularity is not in physical space. To determine the behavior of these functions near the edge, it is evidently enough to examine the behavior of their ϕ-independent parts, namely, $V_1^{(m)}$ and grad $V_1^{(m)}$. We begin by assuming that $0 < |z/a| < \epsilon$ where $\epsilon < r$ is a small positive number. Then, we deform the contour near $t = 0$ in the right member of equation (3.11) into a semi-circle of radius ϵ centered at $t = 0$ and falling below the real t-axis; $|z/a|$ will then be less than $|t|$ everywhere on the contour. Next, we observe, that the integral representation of $V_1^{(m)}$ may be expressed as follows* [cf. equation (3.1)]:

$$(3.21)\quad V_1^{(m)} = r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m-1} \oint_{-1}^{1} \frac{d}{dt}\left[\frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)}\right] \frac{f_1^{(m)}(t)}{ia(z+iat)}dt\ ,$$

since

$$(3.22)\quad \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{ikR^+}}{R^+}\right) = \frac{1}{ia(z+iat)}\frac{d}{dt}\left(\frac{e^{ikR^+}}{R^+}\right)$$

Integrating by parts, we find, in view of 'edge' behavior of $f_1^{(m)}(t)$ [see equations (3.12) and (3.13)] that

---

*A circle has been affixed to the integral sign to distinguish between the present path of integration and that used earlier.

$$(3.23)\quad v_1^{(m)} = -\frac{r^m}{ia}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m-1} \oint_{-1}^{1} \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} \left(\frac{d}{dt}\frac{1}{z+iat}\right) f_1^{(m)}(t)dt \ .$$

Repeating this argument m-1 times, we find that*

$$(3.24)\quad v_1^{(m)} = \left(\frac{ir}{a}\right)^m \oint_{-1}^{1} \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} \left[\frac{d}{dt}\left(\frac{1}{z+iat}\right)\right]^m f_1^{(m)}(t)dt$$

$$= \frac{-1}{\alpha}\left(\frac{ir}{a}\right)^m \oint_{-1}^{1} \frac{d}{dt} \exp\left[ikR^+(r,z;t)\right] \frac{1}{z+iat}\left[\frac{d}{dt}\left(\frac{1}{z+iat}\right)\right]^m f_1^{(m)}(t)dt \ ,$$

where $\alpha = ka$.

Now let T_ρ be a torus whose axis of symmetry is the z-axis and whose equation in any cross section through the z-axis is [see Figure 2]

$$(3.26)\qquad T_\rho: \quad \begin{cases} z = \rho \sin \delta \ , \\ r = a + \rho \cos \delta \ . \end{cases}$$

Here, ρ is the generating circle of T_ρ and δ is the angle measured clockwise from the screen to the ray in the cross section passing through the edge and the point $P(r,z)$. In these coordinates R^+ becomes

---

* The integral appearing in the first equation of (3.24) has essentially the same form as that encountered in reference [1] at a similar stage in the development and could, in fact, be manipulated in a similar manner to obtain the edge behavior. Here, we shall pursue a somewhat different course.

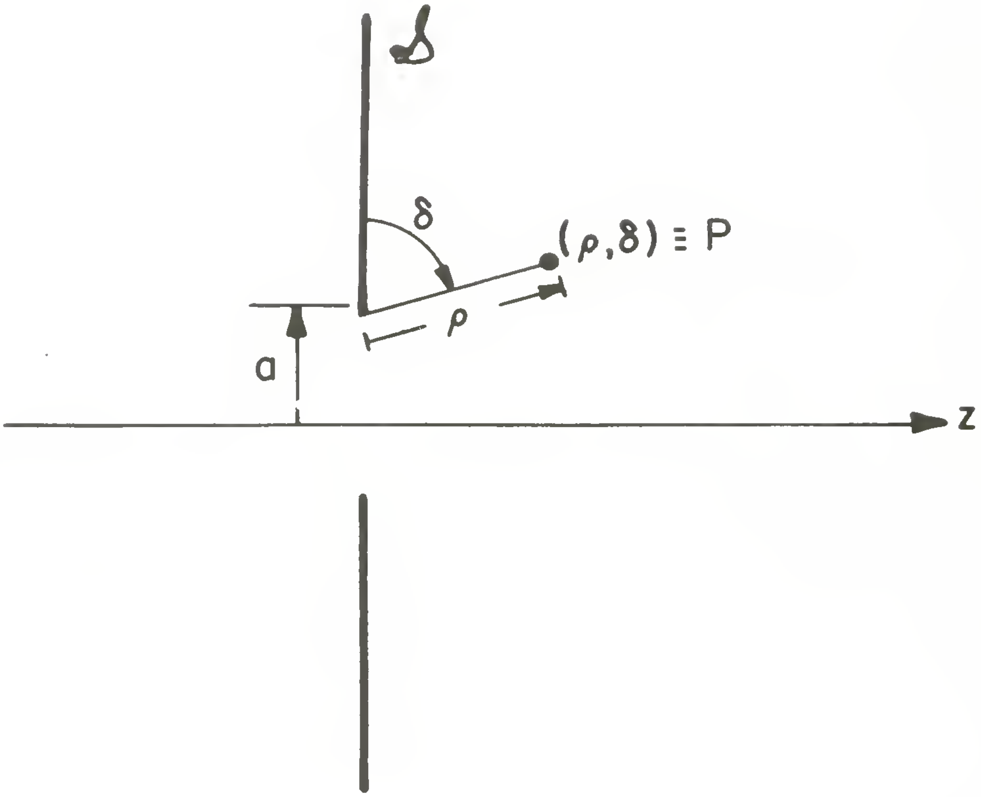

Figure 2

A cross-sectional view of the diffracting screen. The angle δ is measured from the screen clockwise to the ray joining the edge of the screen to the point P; ρ is the distance from the edge to the point P.

$$(3.27) \qquad R^{+} = a\sqrt{1 - t^{2} + 2(\rho/a)t\, e^{i\delta} + (\rho/a)^{2}}\ .$$

Ignoring all but the lowest order terms in (ρ/a), we find that*

$$(3.28) \qquad V_1^{(m)} = \frac{2\sqrt{2}}{i\,a^{m+1}} \frac{d^m f_1^{(m)}(1)}{dt^m} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \sin \delta/2 + \mathcal{O}[(\rho/a)]\ .$$

By similar means, one can ascertain the behavior of $\dfrac{\partial V_1^{(m)}}{\partial \rho}$ and $\dfrac{1}{\rho}\ \dfrac{\partial V_1^{(m)}}{\partial \delta}$. It turns out that

$$(3.29) \qquad (\rho/a)^{\frac{1}{2}} \frac{\partial V_1^{(m)}}{\partial \rho} = \frac{\sqrt{2}}{i\,a^{m+2}} \frac{d^m f_1^{(m)}(1)}{dt^m} \sin(\delta/2) + \mathcal{O}[(\rho/a)^{\frac{1}{2}}]\ ,$$

$$(3.30) \qquad (\rho/a)^{\frac{1}{2}} \frac{1}{\rho} \frac{\partial V_1^{(m)}}{\partial \delta} = \frac{\sqrt{2}}{i\,a^{m+2}} \frac{d^m\, f_1^{(m)}(1)}{dt^m} \cos(\delta/2) + \mathcal{O}[(\rho/a)^{\frac{1}{2}}]\ ,$$

which also follow formally from equation (3.28) on applying the operators $\partial/\partial\rho$ and $\rho^{-1}\ \partial/\partial\delta$.

Equations (3.28) - (3.30) and the remarks made above show that V_1 and grad V_1 satisfy conditions (v) and (vi) of equation (2.3).

Letting z approach zero in equation (3.11), one finds with the aid of equations (3.14) and (3.15) and the oddness of $f_1^{(m)}(t)$ that

---

[*] One might expect the last term in equation (3.25) to contribute a constant term to this expression. However, it is easy to show that this constant is zero owing to the oddness of $f_1^{(m)}(t)$. In the second boundary value problem, however, the corresponding constant term does not vanish [see equation (6.3)].

$$(3.31)\quad V_1^{(m)} = \frac{2r^m}{ia}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \int_{(r/a)}^{1} \frac{\cosh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}}\, f_1^{(m)}(t)dt$$

in the aperture $0 \leqq r < a$. Moreover, recalling that $R^+ = R + iat\cos\theta + \mathcal{O}(R^{-1})$ one infers from equation (3.11) that

$$(3.32)\quad V_1^{(m)} \sim \left(-2\int_0^1 \sinh(\alpha t\cos\theta) f_1^{(m)}(t)dt\right) r^m\left(\frac{1}{R}\frac{d}{dR}\right)^m \frac{e^{ikR}}{R}$$

$$\dot{\sim}\; A_1^{(m)}(\theta)\,\frac{e^{ikR}}{R}, \qquad R \to \infty$$

where

$$(3.33)\quad A_1^{(m)}(\theta) = -2(ik\sin\theta)^m \int_0^1 \sinh(\alpha t\cos\theta) f_1^{(m)}(t)dt$$

is the far field amplitude.

The transmission coefficient $t_1^{(m)}$, associated with the m-th mode, may be expressed as follows:

$$(3.34)\quad t_1^{(m)} = \frac{\dfrac{2\pi}{\epsilon_m}\displaystyle\int_0^{\pi/2} \left|A_1^{(m)}(\theta)\right|^2 \sin\theta\, d\theta}{P_o}\;;\quad \epsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \geqq 1 \end{cases} .$$

The numerator of this expression is the energy flux in the m-th mode that is transmitted to infinity on the shadow side of the screen. The denominator is energy flux in the incident wave alone that is transmitted through the aperture. Here, if $\underset{\sim}{n}$ denotes the outward-directed normal at each point of Σ, then the flux of energy through Σ is taken to be

$$\frac{1}{2ik} \int_{\Sigma} \left(W^* \frac{\partial W}{\partial n} - \frac{W \partial W^*}{\partial n} \right) d\Sigma$$

where W^* is the complex conjugate of W. From the orthogonality of the set $\left\{ \cos m(\phi - \phi_o^{(m)}) \right\}$ $m = 0,1,2,\ldots$, it follows that the transmission coefficient of a sum of modes is the sum of the transmission coefficients of these modes.

4. INTEGRAL EQUATIONS FOR THE $f_1^{(m)}$'s

There remains one condition to be satisfied - the aperture condition as formulated in equation (3.4). It will now be shown that this condition leads to Fredholm integral equations of the second kind for the unknown functions $f_1^{(m)}$, $m = 0,1,2,\dots$. We begin by calculating the limiting value of $\partial V_1^{(m)}/\partial z$. Assuming that $z > 0$, we differentiate both sides of equation (3.11), make use of the identity

$$\frac{\partial M\left[R^{\pm}(r,z;t)\right]}{\partial z} = (z \pm iat)\,\frac{1}{r}\frac{\partial}{\partial r}\,M\left[R^{\pm}(r,z;t)\right]$$

and the oddness of $f_1^{(m)}(t)$ [$M(x)$ is any differentiable function of x], and arrive at the following result:

$$(4.1) \qquad \frac{\partial V_1^{(m)}}{\partial z} = r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1}\int_0^1 \frac{(z+iat)\exp\left[ikR^+(r,z;t)\right]}{R^+(r,z;t)}\,f_1^{(m)}(t)dt$$

$$- \, r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1}\int_0^1 \frac{(z-iat)\exp\left[ikR^-(r,z;t)\right]}{R^-(r,z;t)}\,f_1^{(m)}(t)dt.$$

If we now let z-approach zero, invoke the aperture condition [equation (3.4)], and make use of the limiting behavior of $R^{\pm}(r,z;t)$ [see equations (3.14) and (3.15)], we find, in the aperture, $0 \leqq r < a$, that

$$
\begin{aligned}
(4.2) \qquad u_1^{(m)}(r) &= i\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1} \int_0^{r/a} \left\{ \frac{\exp\left[i\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}} \right. \\
&\qquad \left. + \frac{\exp\left[i\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}} \right\} t f_1^{(m)}(t)\,dt \\
&\quad + \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1} \int_{r/a}^{1} \left\{ \frac{\exp\left[-\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}} \right. \\
&\qquad \left. - \frac{\exp\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}} \right\} t f_1^{(m)}(t)\,dt \\
&= 2i\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1} \int_0^{r/a} \frac{\cos\left[\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}}\, t f_1^{(m)}(t)\,dt \\
&\quad - 2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1} \int_0^{r/a} \frac{\sin\left[\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}}\, t f_1^{(m)}(t)\,dt \\
&\quad - 2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1} \int_{r/a}^{1} \frac{\sinh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}}\, t f_1^{(m)}(t)\,dt.
\end{aligned}
$$

Since $\sin\left[\alpha\sqrt{(r/a)^2-t^2}\right] \Big/ \sqrt{(r/a)^2-t^2}$, $0 \leq t^2 \leq (r/a)^2$ is the continuation of $\sinh\left[\alpha\sqrt{t^2-(r/a)^2}\right] \Big/ \sqrt{t^2-(r/a)^2}$, $t^2 \geq (r/a)^2$, we conclude that for $0 \leq r < a$

$$(4.3)\quad \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1}\int_0^{r/a}\frac{\cos\left[\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}}\,t\dot{f}_1^{(m)}(t)dt$$

$$=\frac{u_1^{(m)}(r)}{2i}+\frac{1}{i}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1}\int_0^1\frac{\sinh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}}\,tf_1^{(m)}(t)dt\ .$$

Now define $g_1^{(o)}(r)$ to be $u_1^{(m)}(r)/2i$ when $m = 0$, and

$$(4.4)\quad g_1^{(m)}(r)=\frac{1}{2i}\int_0^r s_{m-1}ds_{m-1}\int_0^{s_{m-1}} s_{m-2}ds_{m-2}\int_0^{s_{m-2}}\cdots\int_0^{s_1} su_1^{(m)}(s)ds$$

$$+\sum_{\mu=o}^{m-1} d_\mu r^{2\mu}$$

$$=\frac{1}{2^m i(m-1)!}\int_0^r s(r^2-s^2)^{m-1}u_1^{(m)}(s)ds+\sum_{\mu=o}^{m-1} d_\mu r^{2\mu}$$

$$=\frac{r^{2m}}{2^m i(m-1)!}\int_0^1 s(1-s^2)^{m-1}u_1^{(m)}(rs)ds+\sum_{\mu=o}^{m-1} d_\mu r^{2\mu}$$

when $m \geqq 1$.* The d_μ's are constants to be chosen at one's convenience;

---

*Since $u_1^{(m)}(s)$ is, by hypothesis, an even function of s, it follows that $g_1^{(m)}(r)$ is an even function of r. Note also that $g_1^{(m)}(r) = \mathcal{O}[\exp(Ar)]$ since $u_1^{(m)}(s)$ enjoys this property [see Section 2].

they will be used in Section 5 to simplify the expressions for the $g_1^{(m)}(r)$'s associated with plane-wave excitation. For the theory of the present section, one may take them to be zero or equivalently one may absorb them into the constants $c_o, c_1, \ldots, c_{m-1}$ to be introduced presently.

Operating on $g_1^{(m)}(r)$ with $(\frac{1}{r}\frac{\partial}{\partial r})^m$, we evidently have

$$(4.5) \qquad \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m g_1^{(m)}(r) = \frac{1}{2i}\, u_1^{(m)}(r), \qquad m \geqq 1 .$$

From this equation and equation (4.3), it follows directly that

$$(4.6) \qquad \frac{1}{r}\frac{\partial}{\partial r}\int_o^{r/a} \frac{\cos\left[\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}}\, t f_1^{(m)}(t)dt$$

$$= g_1^{(m)}(r) + \sum_{\nu=o}^{m-1} \frac{c_\nu}{\nu!}\, \frac{r^{2\nu}}{a^{2\nu+2}}$$

$$+ \frac{1}{ir}\frac{\partial}{\partial r}\int_o^1 \frac{\sinh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}}\, t f_1^{(m)}(t)dt .$$

The quantities $c_o, c_1, \ldots, c_{m-1}$ are at this point arbitrary constants; they will be fixed later by imposing the 'edge' conditions as formulated in equations (3.12) and (3.13).

Equation (4.6) is of the same form as the corresponding equation for the axially-symmetric case studied in reference [1] [cf. equation (4.7) in [1]] and may be treated in a completely analogous fashion. First, we integrate both side of equation (4.6) from 0 to r and obtain the integral equation

$$(4.7)\quad \int_0^{r/a} \frac{\cos\left[\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}}\, tf_1^{(m)}(t)dt$$

$$= \int_0^r sg_1^{(m)}(s)ds + \sum_{\nu=0}^{m-1} \frac{c_\nu}{2(\nu+1)!} \left(\frac{r}{a}\right)^{2\nu+2}$$

$$+ \frac{1}{i} \int_0^1 \frac{\sinh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}}\, tf_1^{(m)}(t)dt - C_1^{(m)}$$

where

$$(4.8)\quad C_1^{(m)} = \frac{1}{i} \int_0^1 \frac{\sinh \alpha s}{s}\, sf_1^{(m)}(s)ds\ .$$

Next, we introduce the variables

$$(4.9)\qquad \xi = t^2,$$

$$(4.10)\qquad \eta = (r/a)^2,$$

into both sides of equation (4.7) which then assumes the form

$$(4.11)\quad \int_0^\eta \frac{\cos\left[\alpha\sqrt{\eta-\xi}\right]}{\sqrt{\eta-\xi}}\, F_1^{(m)}(\xi)d\xi = H_1^{(m)}(\eta) + J_1^{(m)}(\eta),$$

where

$$(4.12) \qquad F_1^{(m)}(\xi) = f_1^{(m)}(\sqrt{\xi}\,)/2,$$

$$(4.13) \qquad H_1^{(m)}(\eta) = \int_0^{a\sqrt{\eta}} sg_1^{(m)}(s)ds + \frac{1}{2} \sum_{\nu=0}^{m-1} \frac{c_\nu \eta^{\nu+1}}{(\nu+1)!}$$

$$(4.14) \qquad J_1^{(m)}(\eta) = \frac{1}{i} \int_0^1 \frac{\sinh\left[\alpha\sqrt{\xi-\eta}\right]}{\sqrt{\xi-\eta}} F_1^{(m)}(\xi)d\xi - C_1^{(m)} \,.$$

Employing the fact that the Laplace transform of a convolution of two functions is the product of the Laplace transforms, we can solve equation (4.11) for $F_1^{(m)}$ and express the result as follows:

$$(4.15) \qquad F_1^{(m)} = \frac{1}{\pi}\frac{d}{d\eta} \int_0^\eta \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]}{\sqrt{\eta-\mu}} \left[H_1^{(m)}(\mu) + J_1^{(m)}(\mu)\right]d\mu$$

$$= h_1^{(m)}(\eta) + j_1^{(m)}(\eta).$$

Here, $h_1^{(m)}(\eta)$ and $j_1^{(m)}(\eta)$ are defined by

$$(4.16) \qquad h_1^{(m)}(\eta) = \frac{1}{\pi}\frac{d}{d\eta} \int_0^\eta \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]}{\sqrt{\eta-\mu}} H_1^{(m)}(\mu)$$

$$= \frac{1}{\pi}\frac{d}{d\eta} \int_0^\eta \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]}{\sqrt{\eta-\mu}} \left(\int_0^{a\sqrt{\mu}} sg_1^{(m)}(s)ds + \sum_{\nu=0}^{m-1} \frac{c_\nu}{2(\nu+1)!}\, \mu^{\nu+1} \right) d\mu$$

$$(4.17)\qquad j_1^{(m)}(\eta) = \frac{1}{\pi}\frac{d}{d\eta}\int_0^{\eta} \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]}{\sqrt{\eta-\mu}}\, J_1^{(m)}(\mu)\, d\mu$$

$$= \frac{1}{\pi i}\int_0^{1} F_1^{(m)}(\xi)\left[\frac{d}{d\eta}\int_0^{\eta} \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]\sinh\left[\alpha\sqrt{\xi-\mu}\right]}{\sqrt{\eta-\mu}\ \sqrt{\xi-\mu}}\, d\mu\right] d\xi$$

$$- \frac{C_1^{(m)}}{\pi}\cosh\left(\alpha\sqrt{\eta}\right)$$

The expression for $h_1^{(m)}(\eta)$ may be simplified somewhat by (1) observing that $(-2/\alpha)d\left[\sinh(\alpha\sqrt{\eta-\mu})\right] = \left[\cosh(\alpha\sqrt{\eta-\mu})/\sqrt{\eta-\mu}\,\right]d\mu$, (2) integrating by parts and carrying out the indicated differentiation, and finally (3) introducing a new variable of integration, β, where $\mu = \eta\sin\beta$. The result is

$$(4.18)\qquad h_1'^{(m)}(\eta) = \frac{a^2\sqrt{\eta}}{\pi}\int_0^{\pi/2} \cosh(\alpha\sqrt{\eta}\cos\beta)\sin\beta\ g_1^{(m)}(a\sqrt{\eta}\sin\beta)\, d\beta$$

$$+ \frac{1}{\pi}\,.\sum_{\nu=0}^{m-1}\frac{c_\nu}{\nu!}\left(\eta^{\nu+\frac{1}{2}}\int_0^{\pi/2}\cosh(\alpha\sqrt{\eta}\cos\beta)(\sin\beta)^{2\nu+1}d\beta\right)$$

With the aid of Poisson's formula - i.e.

$$(4.19)\qquad J_\ell(w) = \frac{2(\frac{w}{2})^{\ell}}{\sqrt{\pi}\,\Gamma(\ell+\frac{1}{2})}\int_0^{\pi/2}\cos(w\cos\beta)\sin^{2\ell}\beta\, d\beta\ , \qquad \ell > -\tfrac{1}{2}$$

we may write

$$(4.20) \qquad h_1^{(m)}(\eta) = \frac{a^2\sqrt{\eta}}{\pi} \int_0^{\pi/2} \cosh(\alpha\sqrt{\eta}\cos\beta)\sin\beta \; g_1^{(m)}(a\sqrt{\eta}\sin\beta)d\beta$$

$$+ \frac{1}{2\sqrt{\pi}} \sum_{\nu=0}^{m-1} c_\nu \, \eta^{\nu+\frac{1}{2}} \frac{J_{\nu+\frac{1}{2}}(i\alpha\sqrt{\eta})}{(i\alpha\sqrt{\eta}/2)^{\nu+\frac{1}{2}}}$$

The expression for $j_1^{(m)}(\eta)$ may likewise be simplified. Following D.S. Jones*, we (1) make use of the addition formula for hyperbolic functions, (2) introduce obvious changes of variables of integration and (3) carry out the differentiation with respect to η. The final result is

$$(4.21) \qquad j_1^{(m)}(\eta) = \frac{1}{2\pi i} \int_0^1 F_1^{(m)}(\xi) \left\{ \frac{\sinh\left[\alpha(\sqrt{\xi}-\sqrt{\eta})\right]}{\sqrt{\eta}(\sqrt{\xi}-\sqrt{\eta})} + \frac{\sinh\left[\alpha(\sqrt{\xi}+\sqrt{\eta})\right]}{\sqrt{\eta}(\sqrt{\xi}+\sqrt{\eta})} \right\} d\xi$$

$$- C_1^{(m)} \cosh(\alpha\sqrt{\eta}) \Big/ \pi \, .$$

Combining equations (4.15), (4.20) and (4.21), we find:

$$(4.22) \qquad F_1^{(m)}(\eta) = \frac{a^2\sqrt{\eta}}{\pi} \int_0^{\pi/2} \cosh(\alpha\sqrt{\eta}\cos\beta)\sin\beta \; g_1^{(m)}(a\sqrt{\eta}\sin\beta)d\beta \, +$$

---

*See reference [7] pp 10-11.

$$+ \frac{1}{2\sqrt{\pi}} \sum_{\nu=0}^{m-1} c_\nu \, \eta^{\nu+\frac{1}{2}} \frac{J_{\nu+\frac{1}{2}}(i\alpha\sqrt{\eta})}{(i\alpha\sqrt{\eta}/2)^{\nu+\frac{1}{2}}}$$

$$+ \frac{1}{\pi i} \int_0^1 F_1^{(m)}(\xi) \left\{ \frac{\sinh\left[\alpha(\sqrt{\xi} - \sqrt{\eta})\right]}{\sqrt{\eta}(\sqrt{\xi} - \sqrt{\eta})} + \frac{\sinh\left[\alpha(\sqrt{\xi} + \sqrt{\eta})\right]}{\sqrt{\eta}(\sqrt{\xi} + \sqrt{\eta})} \right\} d\xi$$

$$- C_1^{(m)} \cosh(\alpha\sqrt{\eta}) \,/\, \pi \; .$$

Replacing η by t^2 and ξ by s^2 wherever ξ and η appear in equation (4.22) and taking into account the definitions of $C_1^{(m)}$ [equation (4.8)] and $F_1^{(m)}(\xi)$ [equation (4.12)] and the fact that $f_1^{(m)}(t)$ is an odd function of t, we obtain the following integral equation for the function $tf_1^{(m)}(t)$ for $m \geqq 1$:

$$(4.23) \quad tf_1^{(m)}(t) = \frac{2a^2t^2}{\pi} \int_0^{\pi/2} \cosh(\alpha t\cos\beta)\sin\beta \, g_1^{(m)}(at\sin\beta)d\beta$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{\nu=0}^{m-1} c_\nu \, t^{2\nu+2} \frac{J_{\nu+\frac{1}{2}}(i\alpha t)}{(i\alpha t/2)^{\nu+\frac{1}{2}}}$$

$$+ \frac{1}{\pi i} \int_{-1}^{1} \left\{ \frac{\sinh\left[\alpha(t-s)\right]}{t-s} - \frac{\cosh(\alpha t)\sinh(\alpha s)}{s} \right\} sf_1^{(m)}(s)ds \; .$$

When $m = 0$, the sum-term in the right member of this equation is absent. Observe that the Bessel functions in equation (4.23) are actually 'elementary' transcendental functions since

$$\frac{J_{\nu+\frac{1}{2}}(w)}{w^{\nu+\frac{1}{2}}} = \sqrt{\frac{2}{\pi}}\left(-\frac{1}{w}\frac{d}{dw}\right)^{\nu} \frac{\sin w}{w} \quad , \quad \nu = 0,1,2,\ldots \quad .$$

From the evenness* of $g_1^{(m)}(w)$ and $J_{\nu+\frac{1}{2}}(w)/w^{\nu+\frac{1}{2}}$, it follows that the inhomogeneous terms of equation (4.23) are even functions of t. On replacing t by -t and s by -s wherever s and t appear, we find that $-tf_1^{(m)}(-t)$ satisfies the same integral equations as $tf_1^{(m)}(t)$. Thus, assuming merely that the solution of equation (4.23) is unique, we may conclude that $tf_1^{(m)}(t)$ is even and hence that $f_1^{(m)}(t)$ is odd. Furthermore, from the behavior of $g_1^{(m)}(w)$ * and $J_{\nu+\frac{1}{2}}(w)$ for large w, we see that there exists an $N > 0$ such that $f_1^{(m)}(t) = \Theta\left[e^{N|t|}\right]$ when $|t|$ becomes large without bound so that our earlier use of the Laplace transformation is justified.

The above remarks are independent of the particular choice of the c_ν's employed. We shall now show how to fix the c_ν's and how to establish the unique existence of the solution for sufficiently small values of α. For this purpose, we operate on both sides of equation (4.23) with $(t^{-1}\frac{d}{dt})^{\mu}t^{-2}$, $\mu = 0,1,\ldots,m-1$, set $t = 1$ and finally, enforce the 'edge' conditions of equation (3.12). We obtain as a result the system of equations

$$(4.24) \qquad \sum_{\nu=0}^{m-1} c_\nu \, T_\nu^{(\mu)}(\alpha) = I^{(\mu)}(\alpha) + K^{(\mu)}(\alpha), \qquad \mu = 0,1,\ldots(m-1),$$

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*See footnote on page 18.

where

$$(4.25) \qquad T_{\nu}^{(\mu)}(\alpha) = \left(\frac{1}{t}\frac{d}{dt}\right)^{\mu}\left[t^{2\nu}\frac{J_{\nu+\frac{1}{2}}(i\alpha t)}{\left(\frac{i\alpha t}{2}\right)^{\nu+\frac{1}{2}}}\right]\Bigg|_{t=1}$$

$$(4.26) \qquad I^{(\mu)}(\alpha) = \frac{2a^{2}}{\sqrt{\pi}}\left(\frac{1}{t}\frac{d}{dt}\right)^{\mu}\int_{0}^{\pi/2}\cosh(\alpha t\cos\beta)\sin\beta\, g_{1}^{(m)}(at\sin\beta)\,d\beta$$

and

$$(4.27) \qquad K^{(\mu)}(\alpha) = \frac{1}{\sqrt{\pi}\, i}\left(\frac{1}{t}\frac{d}{dt}\right)^{\mu}\frac{1}{t^{2}}$$

$$\times\int_{-1}^{1}\left\{\frac{\sinh[\alpha(t-s)]}{t-s} - \frac{\cosh(\alpha t)\sinh(\alpha s)}{s}\right\} s f_{1}^{(m)}(s)\,ds\Bigg|_{t=1}$$

The elements of the m × m matrix $\left(T_{\nu}^{(\mu)}(\alpha)\right)$ $\mu,\nu = 0,1,\ldots,m-1$, are clearly integral functions of α. Moreover, $\left(T_{\nu}^{(\mu)}(0)\right)$ is a triangular matrix with non- zero elements along its main diagonal - in fact

$$(4.29) \qquad T_{\nu}^{(\nu)}(0) = 2^{\nu}\nu! \qquad \nu = 0,1,\ldots,m-1.$$

Thus, for sufficiently small values of $|\alpha|$, $|\alpha| < \alpha_{0}^{(m)}$ say, $T_{\nu}^{(\nu)}(\alpha)$ has an inverse. This being the case, we may, with the aid of Cramer's rule, write the solution of the system (4.24) in the form

$$(4.29) \qquad c_\nu = \frac{\sum_{\mu=0}^{m-1} I^{(\mu)}(\alpha) A_\nu^{(\mu)}(\alpha)}{\left|T_\nu^{(\mu)}(\alpha)\right|} + \frac{\sum_{\mu=0}^{m-1} K^{(\mu)}(\alpha) A_\nu^{(\mu)}(\alpha)}{\left|T_\nu^{(\mu)}(\alpha)\right|},$$

$$0 \leqq |\alpha| \leqq \alpha_0^{(m)},$$

where $|T_\nu^{(\mu)}(\alpha)|$ is the determinant of $\left(T_\nu^{(\mu)}(\alpha)\right)$ and $A_\nu^{(\mu)}(\alpha)$ is the cofactor of the element $T_\nu^{(\mu)}(\alpha)$. The c_ν's evidently involve two types of terms, those which involve $f_1^{(m)}$ [through the $K^{(\mu)}$'s] and those which depend upon the $g_1^{(m)}$'s. If we (1) substitute the above expression for c_ν into equation (4.23), (2) group the terms which depend on $f_1^{(m)}$ with the terms involving the kernel and finally, (3) group the terms which are independent of $f_1^{(m)}$ with the original inhomogeneous term, we arrive at a new integral equation whose essential features may be described as follows: The new inhomogeneous term shares with the old the property of being a regular function of t in the disc $0 \leqq |t| \leqq 1 + \Delta/a$. In effect, the domain of regularity of this term is controlled by that of $g_1^{(m)}(t)$ [see footnote on page 23]. Furthermore, the new kernel shares with the original kernel the property of being $O(\alpha^3)$ when α is small. This follows directly from equation (4.27) on representing the terms within the curly brackets by their power series. We may therefore assert that the new integral equation has a unique continuous (Neumann series) solution for sufficiently small values of α. By making $\alpha_o^{(m)}$ introduced above smaller, if necessary, we may without loss of generality assume that the Neumann series converges whenever $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$. With the unique existence of a continuous $f_1^{(m)}(t)$ established for $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$, we can infer from equations (4.29) and (4.25) - (4.27) that the c_ν's are regular functions of α, $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$ and that $f_1^{(m)}(t)$ is regular in the disc $0 \leqq |t| \leqq 1 + \Delta/a$. The first remark follows directly from equations (4.29) and (4.24) - (4.27). To prove the second, it is enough to observe that (1) the new kernel is an entire function of t and (2) the inhomogeneous term is regular for all t in the disc $0 \leqq |t| \leqq 1 + \Delta/a$. Thus at least for $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$ the unique existence for the desired type of $f_1^{(m)}(t)$ is established.

In applications, it is often the case that the first term in the right member of equation (4.23) is an analytic function of the two complex variables α and t in the region $\mathcal{R}$ defined by $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$, $0 \leqq |t| \leqq 1 + \Delta/a$. Since the kernel is analytic for all (α,t), each term in the Neumann series referred to above is an analytic function of (α,t) in $\mathcal{R}$. Moreover, as the convergence is uniform for (α,t) in $\mathcal{R}$, we may conclude that the limit function - i.e., $f_1^{(m)}(t)$, is analytic in the interior of $\mathcal{R}$ and may be expressed, in this domain, as a power series in α with power-series coefficients in t. This fact suggests the following scheme for calculating the $f_1^{(m)}$'s. Start with equation (4.23) and write each c_ν as a power series in α - i.e., write

$$c_\nu = \sum_{\mu=0}^{\infty} c_\nu^{(\mu)}\alpha^\mu.$$

Next, set

$$f_1^{(m)}(t) = \sum_{\mu=0}^{\infty} A_\mu^{(m)}(t)\alpha^\mu,$$

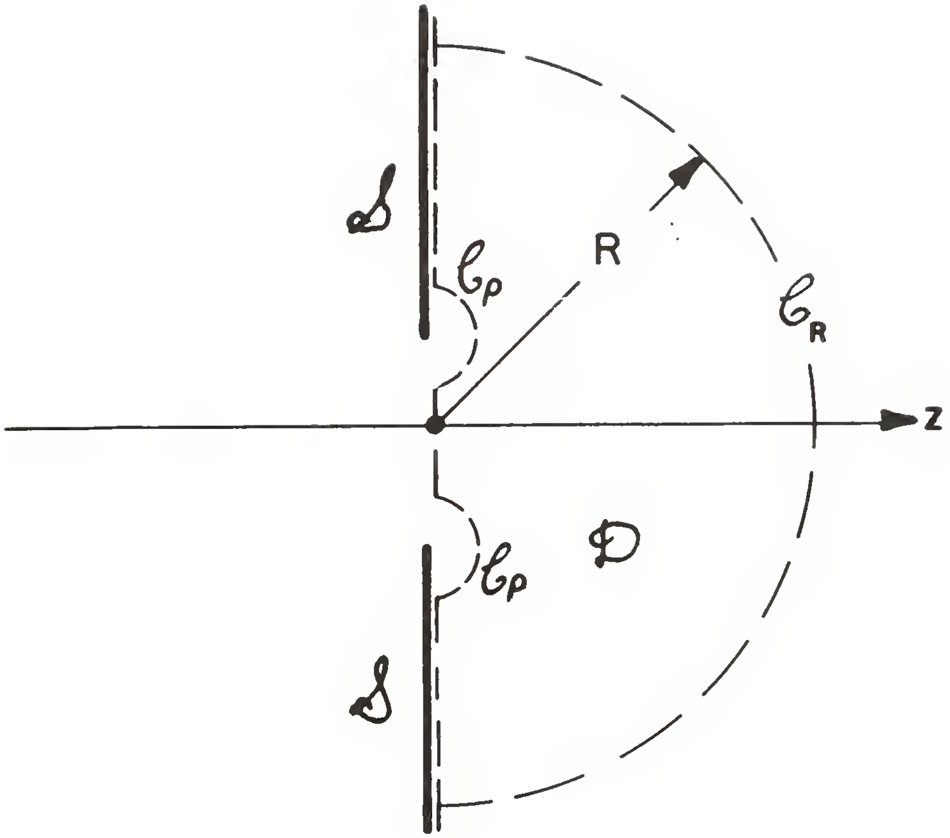

Figure 3

A cross-sectional view of the surfaces (shown dashed) that bound the domain $\mathcal{D}$. $\mathcal{C}_R$ and $\mathcal{C}_\rho$ are meridianal semi-circles cut out of a hemisphere and torus whose equatorial planes coincide with the plane of the screen $\mathcal{S}$. Both torus and hemisphere are centered at O.

substitute this expression into equation (4.23) and equate like powers of α. The result is a sequence of integral relations expressing $A_\mu^{(m)}(t)$ in terms of the $c_\nu^{(\mu)}$'s and integral terms involving $A_\mu^{(m)}$'s with lower μ-indices. Since the kernels of the integral terms are polynomials in s and t, it follows that $A_\mu^{(m)}(t)$ is a polynomial in t. Moreover, as the first few $A_\mu^{(m)}$'s are easy to calculate explicitly, it follows that the $A_\mu^{(m)}(t)$ may be calculated recursively. The final step, once the $A_\mu^{(m)}$'s are known, is to fix the $c_\nu^{(\mu)}$'s by imposing the 'edge' conditions of equation (3.12). If terms of order higher than α^8, say, are ignored, this procedure seems to be a very efficient method for calculating $f_1^{(m)}(t)$. We have made use of it in Sections 5 and 7 although, in the problems considered there, explicit formulas for the c_ν's are easily found.

Whenever a solution $f_1^{(m)}(t)$ of equation (4.23) with the required properties exists, whether as a Neumann series or in terms of the Fredholm resolvent, the integral representations of Section 3 furnish a solution V_1 of the aperture problem and equation (3.11) gives the form of the edge behavior. It is easily shown that V_1 is unique in the class of solutions having the properties listed earlier [in the neighborhood of equation (3.3)] if the radiation condition is strengthened by requiring k to have a small positive imaginary part* $\mathrm{Im}\,(k) > 0$. For, if V_1' is another solution then $v = V_1 - V_1'$ enjoys all the properties of V_1 except that $\partial v/\partial z$ vanishes in the aperture. Consider the integral

$$\int_{\mathcal{D}} \nabla(v^* \cdot \nabla v)\,dW \;;$$

here v^* denotes the complex conjugate of v; $\mathcal{D}$ is the domain contained within the surface shown in Figure 3 (in cross section) and dW is the volume element. Employing (1) Green's Theorem, (2) the vanishing of v and $\partial v/\partial z$ on the aperture and the screen, respectively, and (3) the fact that v satisfies the wave equation and the radiation condition, we find, on letting ρ and R^{-1} approach zero, that

---

*For a uniqueness proof that does not make use of this added assumption and which, in fact, weakens several others listed earlier, see the forthcoming report [13] by L. Levine.

$$\int_{\mathcal{D}} (|\nabla v|^2 - k^2|v|^2)dW = 0.$$

From this and the fact that Im (k) $\neq$ 0, it follows easily that v vanishes everywhere - i.e., that $V_1 = V_1'$.

This proof evidently applies when the incident wave contains more than one mode; however, the total number of incident modes must be finite. When an infinite number of modes are present, one encounters problems relating to convergence. These require for their resolution knowing, among other things, how $\alpha_0^{(m)}$ and $\max\limits_{0 \leqq t \leqq 1} |f_1^{(m)}(t)|$ depend on m as m approaches infinity. It appears that complete success in dealing with such problems hinges on obtaining simple explicit formulas for the c_ν's, possibly by enforcing the 'edge' conditions of equation (3.12) at an earlier stage and by deriving the integral equation in a different fashion. Results on this question will be presented in a future report of this series.

Finally, it should be mentioned that it is possible to construct the solution of equation (4.23) in terms of the eigenfunctions of the (symmetric) kernel $\sinh[\alpha(t-s)]/(t-s)$. As D.S. Jones[7] has already pointed out, these eigenfunctions are spheroidal functions.* However, it appears that a solution in which $f_1^{(m)}(t)$ is so expressed would have little advantage over that obtained by employing spheroidal wave functions at the very beginning.

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*In a private communication to J. Bazer and A. Brown, C.J. Bouwkamp has shown how to convert their solution of the second boundary-value problem to his original solution in terms of spheroidal wave functions.

5. PLANE WAVE EXCITATION - THE FIRST BOUNDARY VALUE PROBLEM

In the present case, the term involving $g_1^{(m)}$ in the integral equation (4.23) reduces to a simple expression. To derive this expression, it seems that the most efficient procedure consists in backtracking somewhat to a suitable stage in the derivation of equation (4.23). Let $I_1^{(m)}(\eta)$ denote the first term in the right member of equation (4.22). Then from (4.16), it follows that

$$(5.1) \qquad I_1^{(m)}(\eta) = \frac{1}{\pi} \frac{d}{d\eta} \int_0^{\eta} \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]}{\sqrt{\eta-\mu}} \left(\int_0^{a\sqrt{\mu}} s g_1^{(m)}(s) ds \right)$$

Choosing the arbitrary constants which appear in the definition of $g_1^{(m)}(s)$ [see equation (4.4)] in an obvious manner, we find, in view of equation (2.13), that

$$g_1^{(m)}(s) = \frac{k \cos\gamma\, \epsilon_m (-i)^m}{2(k \sin\gamma)^m} J_0(ks \sin\gamma).$$

Combining equations (5.1) and (5.2), we then have

$$(5.2) \qquad I_1^{(m)}(\eta) = \frac{\alpha \cos\gamma\, \epsilon_m (-i)^m}{2\pi (k \sin\gamma)^{m+1}} \frac{d}{d\eta} \int_0^{\eta} \frac{\cosh\left[\alpha\sqrt{\eta-\mu}\right]}{\sqrt{\eta-\mu}}$$

$$\times J_1(\alpha \sqrt{\mu} \sin\gamma)\ \sqrt{\mu}\, d\mu\ .$$

The η-dependent term in the right member is evidently the derivative of a convolution. Taking the Laplace transform of both sides, simplifying the transforms of the right member and then taking inverse transforms, we get

$$(5.3)\qquad I^{(m)}(\eta) = \frac{\alpha\epsilon_m(-i)^m}{2\pi(k\sin\gamma)^m}\sinh\left[\alpha\sqrt{\eta}\cos\gamma\right] .$$

This, then, is the first term in the right member of equation (4.22), which, on making the substitutions $\eta = t^2$ and $\xi = s^2$, becomes

$$(5.4)\qquad tf_1^{(m)}(t) = \frac{\alpha\epsilon_m(-i)^m}{\pi(k\sin\gamma)^m}\, t\sinh\left[\alpha t\cos\gamma\right]$$

$$+ \frac{\sqrt{\pi}}{2}\sum_{\nu=0}^{m-1} c_\nu\, t^{2\nu+2}\, \frac{J_{\nu+\frac{1}{2}}(i\alpha t)}{(i\alpha t/2)}$$

$$+ \frac{1}{\pi i}\int_{-1}^{1}\left\{\frac{\sinh\left[\alpha(t-s)\right]}{(t-s)} - \frac{\cosh(\alpha t)\sinh(\alpha s)}{s}\right\} sf_1^{(m)}(s)ds.$$

Calculating $tf_1^{(0)}$, $tf_1^{(1)}$ and $tf_1^{(2)}$ according to the scheme outlined in the preceding section, we find that

$$(5.5)\qquad f_1^{(0)} = \frac{(t\cos\gamma)\alpha^2}{\pi k}\left\{1 + \frac{t^2\cos^2\gamma}{3!}\alpha^2 + \frac{2i}{9\pi}\alpha^3 + \frac{\cos^4\gamma}{5!}t^4\alpha^4\right.$$

$$+ i\left[\left(\frac{1}{75\pi} + \frac{\cos^2\gamma}{45\pi}\right) + \frac{t^2}{45\pi}\right]\alpha^5 \quad +$$

$$+ \left[\frac{-4}{81\pi^2} + \frac{\cos^6\gamma}{7!} t^6 \right] \alpha^6 + \mathcal{O}(\alpha^7) \Bigg\} \quad ,$$

$$(5.6) \qquad f_1^{(1)} = - \frac{\sin 2\gamma}{\pi i k^2} t(1-t^2)\alpha^4 \left\{ \frac{1}{3!} + \left[- \frac{1}{(3!)^2} + \frac{(1+t^2)(1+\cos^2\gamma)}{5!} \right] \alpha^2 \right.$$

$$+ \left[\frac{1+\cos^2\gamma+\cos^4\gamma}{7!} (1+t+t^4) - \frac{1+t^2}{3!5!} + \left(\frac{1}{3!^2} - \frac{1+\cos^2\gamma}{5!} \right) \right] \alpha^4$$

$$\left. + \mathcal{O}(\alpha^6) \right\} \quad ,$$

$$(5.7) \qquad f_1^{(2)} = \frac{-2\cos\gamma\sin^2\gamma}{\pi k^3} t(1-t^2)^2\alpha^6 \left\{ \frac{1}{120} + \left[\frac{(2+\cos^2\gamma)(t^2+2)}{7!} - \frac{1}{5.5!} \right] \alpha^2 \right.$$

$$\left. + \mathcal{O}(\alpha^4) \right\} \quad .$$

Let $\epsilon = \epsilon(r)$ be defined by

$$(5.8) \qquad \epsilon = \sqrt{1 - (r/a)^2} \quad , \qquad\qquad 0 \leqq r \leqq a \; .$$

With the aid of equation (3.31), we obtain the following expressions for $V_1^{(m)}(r,0)$, $m = 0,1,2$ in the aperture:*

---

*The total aperture field is $\cos m\phi \; V_1^{(m)}(r,0)$ (cf. equation (3.11)).

$$
\begin{aligned}
(5.9)\quad V_1^{(o)}(r,0) = \frac{-2i\epsilon\cos\gamma}{\pi}\,\alpha \Bigg\{ & 1 + \left[\frac{\cos^2\gamma}{6} + \left(\frac{1}{6} - \frac{\cos^2\gamma}{9}\right)\epsilon^2\right]\alpha^2 + \frac{2i}{9\pi}\,\alpha^3 \\
& + \frac{\alpha^4}{5!}\left[\cos^4\gamma + \frac{2}{3}\cos^2\gamma(5-2\cos^2\gamma)\epsilon^2 + \left(1 + \frac{8}{15}\cos^4\gamma - \frac{4}{3}\cos^2\gamma\right)\epsilon^4\right] \\
& + \left[\frac{i}{45\pi}\left(\frac{8}{5} + \cos^2\gamma\right) + \frac{i}{45\pi}\,\epsilon^2\right]\alpha^5 \\
& + \Bigg[\left(\frac{\cos^6\gamma}{7!} - \frac{4}{81\pi^2}\right) + \left(\frac{\cos^4\gamma}{720} - \frac{2\cos^6\gamma}{7!}\right)\epsilon^2 \\
& + \left(\frac{8}{5}\,\frac{\cos^6\gamma}{7!} - \frac{\cos^4\gamma}{900} + \frac{\cos^2\gamma}{6!}\right)\epsilon^4 \\
& + \frac{1}{7!}\left(1 - 2\cos^2\gamma + \frac{8}{5}\cos^4\gamma - \frac{16}{35}\cos^6\gamma\right)\epsilon^6\Bigg]\alpha^6 \\
& + \quad \mathcal{O}(\alpha^7)\Bigg\} \quad ,
\end{aligned}
$$

$$
(5.10)\quad V_1^{(1)} = \frac{2}{3}\,\frac{\sin 2\gamma}{\pi}\left(\frac{r}{a}\right)\alpha^2\epsilon\left\{1 + \frac{\alpha^2}{5}\left[-\frac{1}{3} + \frac{\cos^2\gamma}{2} + \left(\frac{1}{2} - \frac{\cos^2\gamma}{3}\right)\epsilon^2\right] + \quad \mathcal{O}(\alpha^4)\right\} \quad ,
$$

$$
(5.11)\quad V_1^{(2)} = \frac{4i\cos\gamma\sin^2\gamma}{15\pi}\left(\frac{r}{a}\right)^2\epsilon\alpha^3 + \quad \mathcal{O}(\alpha^5).
$$

From equation (3.33), we obtain the following expressions of the ϕ-independent parts of the far-field amplitudes:

$$(5.12)\quad A_1^{(o)}(\theta) = -\frac{2a\cos\gamma\cos\theta}{3\pi}\,\alpha^2\left\{1+\frac{1}{10}\left[\cos^2\theta+\cos^2\gamma\right]\alpha^2+\frac{2i}{9\pi}\,\alpha^3\right.$$

$$\left.+\frac{1}{280}\left[\cos^4\theta+\frac{10\cos^2\theta\cos^2\gamma}{3}+\cos^4\gamma\right]\alpha^4+\mathcal{O}(\alpha^6)\right\}\ ,$$

$$(5.13)\quad A_1^{(1)}(\theta) = -\frac{\sin2\gamma\sin2\theta}{45\pi}\,\alpha^4\left\{1+\frac{1}{14}\left[\frac{4}{3}-\cos^2\gamma+\cos^2\theta\right]\alpha^2\right.$$

$$+\frac{1}{28}\left[\frac{356}{15}+\frac{97}{5}\cos^2\gamma+18\cos^4\gamma+\frac{1}{180}\left(32-28\cos^2\gamma\right)\cos^2\theta\right.$$

$$\left.\left.+\frac{\cos^4\theta}{18}\right]\alpha^4+O(\alpha^6)\right\}\ :$$

$$(5.14)\quad A_1^{(2)}(\theta) = -\frac{4a\cos\gamma\sin^2\gamma\cos\theta\sin^2\theta}{1575\pi}\,\alpha^6$$

$$\times\left\{1+\frac{1}{9}\left[\left(-\frac{4}{5}+\frac{\cos^2\gamma}{2}\right)+\frac{\cos^2\theta}{2}\right]\alpha^2+\mathcal{O}(\alpha^4)\right\}\ .$$

The corresponding transmission coefficients are:

$$(5.15)\quad t_1^{(o)} = \frac{8\cos\gamma}{27\pi^2}\,\alpha^4 \left\{1 + \frac{1}{5}\left[\frac{3}{5} + \cos^2\gamma\right]\alpha^2 + \frac{1}{175}\left[\frac{9}{7} + \frac{23}{5}\cos^2\gamma + 3\cos^4\gamma\right]\alpha^4 + \mathcal{O}(\alpha^6)\right\} ,$$

$$(5.16)\quad t_1^{(1)} = \frac{32\sin^2\gamma\cos\gamma}{15(45\pi)^2}\,\alpha^8 \left\{1 + \left[\frac{37-21\cos^2\gamma}{147}\right]\alpha^2 + \left[\frac{285,767}{158,760} + \frac{77}{1470}\cos^2\gamma + \frac{253}{196}\cos^4\gamma\right]\alpha^4 + \mathcal{O}(\alpha^6)\right\} ,$$

$$(5.17)\quad t_1^{(2)} = \frac{128\cos\gamma\sin^4\gamma}{(105)(1575\pi)^2}\,\alpha^{12} \left\{1 + \left[\frac{2}{15} + \frac{\cos^2\gamma}{9}\right]\alpha^2 + \mathcal{O}(\alpha^4)\right\}$$

It may be mentioned, that it is merely tedious, not difficult to obtain higher powers of α in each of the above expressions and to calculate the corresponding expressions for $m = 3$. The same applies to the corresponding results listed in Section 7.

In Appendix I, it is shown that[*] [**]

$$(5.18)\quad f_1^{(m)}(t) = \frac{\epsilon_m(-i)^m\cos\gamma\sin^m\gamma\, a^{m+1}}{(2m+1)!\,\pi}\, t(1-t^2)^m \alpha^{m+1}\left[1 + \mathcal{O}(\alpha)\right]$$

for any $m \geqq 0$. With the aid of equations (3.31), (3.33) and (3.34), we then find that

---

[*] The quantity ϵ_m is defined in equation (2.11).

[**] The order estimates in the following expressions are conservative.

$$(5.19)\quad V_1^{(m)} = \frac{-2^{m+1}\epsilon_m i^{m+1} m!\cos\gamma\sin^m\gamma}{(2m+1)!\pi}\left(\frac{r}{a}\right)^m \sqrt{1-\left(\frac{r}{a}\right)^2}\,\alpha^{m+1}\left[1+\mathcal{O}(\alpha)\right],$$

$$(5.20)\quad A_1^{(m)}(\theta) = \frac{-2^{m+2} m!(m+1)!a\epsilon_m(\sin\theta\sin\gamma)^m\cos\theta\cos\gamma}{(2m+1)!(2m+3)!\pi}\,\alpha^{2m+2}\left[1+\mathcal{O}(\alpha)\right],$$

and

$$(5.21)\quad t_1^{(m)} = \frac{\epsilon_m 2^{6m+6}(m!)^3[(m+1)!]^3\cos\gamma\sin^{2m}\gamma}{\pi^2[(2m+1)!]^2[(2m+3)!]^3}\,\alpha^{4m+4}\left[1+\mathcal{O}(\alpha)\right].$$

In the special case of normal incidence ($\gamma = 0$) our formulas for $V_1^{(o)}(r,0)$ and $t_1^{(o)}$ are in agreement with those given by C.J. Bouwkamp [see reference [2] pp. 64 and 72]. We have not succeeded in locating the corresponding expressions for $\gamma \neq 0$. Our results for the higher harmonics $m \geq o$ seem to be new. Observe that several factors combine to make $|V_1^{(m)}|$, $|A_1^{(m)}\theta)|$ and $|t_1^{(m)}|$ small when m is large. First, there is the numerical factor involving the factorials of m. Next, there is the power of α and, finally, there is the factor involving powers of $\sin\gamma$.

6. THE SECOND BOUNDARY-VALUE PROBLEM

We represent the scattered wave $V_2(r,\phi,z)$ in the half-space $z > 0$ as follows:

$$(6.1)\quad V_2 = \cos m(\phi-\phi_o^{(m)})V_2^{(m)}(r,z) =$$

$$= \cos m(\phi-\phi_o^{(m)})\left[r^m\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \int_{-1}^{1} \frac{e^{ikR^+(r,z;t)}}{R^+(r,z;t)} f_2^{(m)}(t)dt\right],$$

$$m = 0,1,2,\ldots .$$

For each m, $f_2^{(m)}(t)$ is required to be (1) an even function of t, (2) a regular function of t in the (closed) circular domain $0 \leqq t \leqq 1 + \Delta/a$ $\Delta > 0$, and (3) to satisfy the 'edge' conditions

$$(6.2)\quad \left.\frac{d^k f_2^{(m)}(t)}{dt^k}\right|_{t=1} = \left.\frac{d^k f_2^{(m)}(t)}{dt^k}\right|_{t=-1} = 0\ , \quad k = 0,1,\ldots,m-1.$$

At points in the aperture, the only points in physical space where the integrand can be singular, V_2 and its derivatives are defined as the limits approached as z approaches zero through positive values.

Repeating the argument of Section 3, we find that conditions (i), (ii) and (iii) [see equations (2.3) and (2.5)] are satisfied. Applying the method presented in Section 3, we find the edge behavior to be:

$$(6.3) \qquad V_2^{(m)} = u_2^{(m)}(1) - \frac{2\sqrt{2}}{a^{m+1}} \frac{d^m f_2^{(m)}(1)}{dt^m} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \cos(\delta/2) + \mathcal{O}[(\rho/a)] ,$$

$$(6.4) \qquad (\rho/a)^{\frac{1}{2}} \frac{\partial V_2^{(m)}}{\partial \rho} = -\frac{\sqrt{2}}{a^{m+2}} \frac{d^m f_2^{(m)}(1)}{dt^m} \cos(\delta/2) + \mathcal{O}\left[(\rho/a)^{\frac{1}{2}}\right] ,$$

$$(6.5) \qquad (\rho/a)^{\frac{1}{2}} \frac{1}{\rho} \frac{\partial V_2^{(m)}}{\partial \delta} = \frac{\sqrt{2}}{a^{m+2}} \frac{d^m f_2^{(m)}(1)}{dt^m} \sin(\delta/2) + \mathcal{O}\left[(\rho/a)^{\frac{1}{2}}\right] .$$

The variables ρ and δ are those introduced earlier in equation (3.26) [see Figure 2]. These results imply that conditions (v) and (vi) of equation (2.3) are satisfied.

From equation (6.1) and equations (3.14), (3.15), (3.22), we obtain the following expressions for $\partial V_2/\partial z$ (in the aperture) and the far field

$$(6.6) \qquad \frac{\partial V_2(r,\phi,0)}{\partial z} = 2\cos m(\phi-\phi_0^{(m)})\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m+1} \int_{r/a}^{1} \frac{\cosh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}} t f_2^{(m)}(t)dt ,$$

$$(6.7) \qquad V_2 \sim \cos m(\phi-\phi_0^{(m)}) A_2^{(m)}(\theta) \frac{e^{ikR}}{R} , \qquad R = \sqrt{r^2+z^2} \to \infty ,$$

where

$$(6.8) \qquad A_2^{(m)}(\theta) = 2(ik\sin\theta)^m \int_0^1 \cosh(\alpha t\cos\theta) f_2^{(m)}(t)dt , \quad m = 0,1,\ldots$$

The transmission coefficient for the m^{th} mode is

$$(6.9) \qquad t_2^{(m)} = \frac{\dfrac{2\pi}{\epsilon_m} \displaystyle\int_0^{\pi/2} \left| A_2^{(m)}(\theta) \right|^2 \sin\theta d\theta}{P_o} , \quad m = 0,1,\dots ,$$

where P_o is the power in the incident wave alone that is transmitted through the aperture [cf. Section 3].

Enforcing the aperture condition, we find for each $m \geqq 0$, the following analog of equation (4.6):

$$(6.10) \qquad \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \int_0^{r/a} \frac{\cos\left[\alpha\sqrt{(r/a)^2-t^2}\right]}{\sqrt{(r/a)^2-t^2}} f_2^{(m)}(t)dt$$

$$= \frac{a}{2} u_2^{(m)}(r) + \frac{1}{i}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m \int_0^1 \frac{\sinh\left[\alpha\sqrt{t^2-(r/a)^2}\right]}{\sqrt{t^2-(r/a)^2}} f_2^{(m)}(t)dt.$$

Proceeding as in Section 4, we are led to the integral equation

$$(6.11) \qquad f_2^{(m)}(t) = \frac{2}{\pi}\frac{d}{dt}\left[t\int_0^{\pi/2} \cosh\left[\alpha t\cos\beta\right] g_2^{m}(\alpha t\sin\beta)\sin\beta d\beta\right]$$

$$+ \frac{2}{\sqrt{\pi}} \sum_{\nu=0}^{m-1} c_\nu t^{2\nu} \frac{J_{\nu-\frac{1}{2}}(i\alpha t)}{\left(\frac{i\alpha t}{2}\right)^{\nu-\frac{1}{2}}} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh\left[\alpha(t-s)\right]}{t-s} f_2^{(m)}(t)dt ,$$

where

$$(6.12) \qquad g_2^{(o)}(r) = \frac{au_2^{o}(r)}{2}$$

when m = 0, and

$$(6.13) \quad g_2^{m}(r) - \sum_{\mu=o}^{m-1} d_\mu r^{2\mu} = \int_0^r s_{m-1} ds_{m-1} \int_0^{s_{m-1}} s_{m-2} ds_{m-2} \cdots \int_0^{s_1} s g_2^{(o)}(s) ds$$

$$= \frac{a}{2^m(m-1)!} \int_0^r s(r^2-s^2)^{m-1} u_2^{(m)}(s) ds$$

$$= \frac{ar^{2m}}{2^m(m-1)!} \int_0^1 s(1-s^2)^{m-1} u_2^{(m)}(rs) ds ,$$

when $m \geqq 1$. The d_μ's are constants to be chosen at one's convenience. By arguments similar to those employed in Section 4, one can establish the unique existence of the desired type of $f_2^{(m)}$'s for sufficiently small values of α. In addition, the uniqueness proof for the total wave function and the scheme outlined at the end of Section 4 for calculating the $f_1^{(m)}$'s as power series in α applies with little change to the present case.

7. PLANE WAVE EXCITATION - THE SECOND BOUNDARY-VALUE PROBLEM

Paralleling the argument of Section 5, we find the following integral equation for $f_1^{(m)}(t)$.

$$(7.1)\quad f_2^{(m)}(t) = \frac{a(-i)^m \epsilon_m}{\pi(k\sin\gamma)^m}\cosh(\alpha t\cos\gamma) + \frac{2}{\sqrt{\pi}}\sum_{\nu=0}^{m-1} c_\nu t^{2\nu}\,\frac{J_{\nu-\frac{1}{2}}(i\alpha t)}{(i\alpha t/2)^{\nu-\frac{1}{2}}}$$

$$+ \frac{1}{\pi i}\int_{-1}^{1}\frac{\sinh[\alpha(t-s)]}{(t-s)}\,f_2^{(m)}(t)dt\,, \qquad m = 0,1,\dots\,.$$

Calculating the $f_2^{(m)}$'s by the method outlined in the end of Section 4, we obtain the following expressions for $f_2^{(0)}$, $f_2^{(1)}$ and $f_2^{(2)}$:*

$$(7.2)\quad f_2^{(0)}(t) = \frac{a}{\pi}\left\{1 - \frac{2i}{\pi}\alpha + \left[-\frac{4}{\pi^2} + \frac{t^2}{2}\cos^2\gamma\right]\alpha^2\right.$$

$$- \frac{i}{\pi}\left[\frac{1+3\cos^2\gamma}{9} - \frac{8}{\pi^2} + \frac{1}{3}t^2\right]\alpha^3$$

$$\left. + \left[\frac{-2\pi^2 + 48 - 2\pi^2\cos^2\gamma}{3\pi^4} - \frac{2}{3\pi^2}t^2 + \frac{\cos^4\gamma}{24}t^4\right]\alpha^4 + \mathcal{O}[\alpha^6]\right\}$$

$$(7.3)\quad f_2^{(1)}(t) = \frac{a^2\sin\gamma(1-t^2)}{\pi i}\,\alpha\left\{1 - \frac{5-\cos^2\gamma - t^2(1+\cos^2\gamma)}{12}\,\alpha^2 + \mathcal{O}[\alpha^3]\right\}$$

---

*The radii of convergence for these power series in α are not known. C.H. Yang[14] has shown that it is at least 1/2 in the case m = 0.

$$(7.4)\quad f_2^{(2)}(t) = -\frac{a^3\sin^2\gamma(1-t^2)^2}{12\pi}\,\alpha^2\left\{1+\mathcal{O}(\alpha)\right\}.$$

The expressions for $(\partial V_2^{(m)}(r,z))/\partial z$ in the aperture are accordingly [cf. equation (6.1) and (6.6)]*

$$(7.5)\quad \left.\frac{\partial V_2^{(o)}}{\partial z}\right|_{z=o} = -\frac{2}{\pi a\epsilon}\left\{1-\frac{2i}{\pi}\alpha-\frac{1}{2}\left[\frac{8-\pi^2\cos^2\gamma}{\pi^2}-(1-2\cos^2\gamma)\epsilon^2\right]\alpha^2\right.$$

$$-\frac{1}{3\pi i}\left[\frac{72-4\pi^2-3\pi^2\cos^2\gamma}{3\pi^2}-\epsilon^2\right]\alpha^3$$

$$-\frac{1}{2}\left[\frac{32\pi^2-384+16\pi^2\cos^2\gamma-\pi^4\cos^4\gamma}{12\pi^4}+\frac{8-3\pi^2\cos^2\gamma+2\pi^2\cos^4\gamma}{6\pi^2}\epsilon^2\right.$$

$$\left.\left.-\frac{3-12\cos^2\gamma+8\cos^4\gamma}{36}\epsilon^4\right]\alpha^4+\mathcal{O}(\alpha^5)\right\},$$

$$(7.6)\quad \left.\frac{\partial V_2^{(1)}}{\partial z}\right|_{z=o} = \frac{4\sin\gamma}{\pi i(a\epsilon)}\left(\frac{r}{a}\right)\alpha\left\{1+\frac{1}{4}\left[\frac{(2\cos^2\gamma-4)}{3}+(2-4\cos^2\gamma)\epsilon^2\right]\alpha^2+\mathcal{O}(\alpha^3)\right\}$$

$$(7.7)\quad \left.\frac{\partial V_2^{(2)}}{\partial z}\right|_{z=o} = \frac{4\sin^2\gamma}{3\pi(a\epsilon)}\left(\frac{r}{a}\right)^2\alpha^2\left\{1+\mathcal{O}(\alpha)\right\},$$

where again $\epsilon = \sqrt{1-(r/a)^2}$.

The corresponding far field ampltiudes are [see equation (6.8)]:

$$(7.8)\quad A_2^{(o)}(\theta) = \frac{2a}{\pi}\left\{1 + \frac{2}{\pi i}\,\alpha + \left[\frac{-24+\pi^2\cos^2\gamma+\pi^2\cos^2\theta}{6\pi^2}\right]\alpha^2\right.$$

$$\left. - i\left[\frac{2\pi^2-72+3\pi^2\cos^2\gamma+3\pi^2\cos^2\theta}{9\pi^3}\right]\alpha^3 + \mathcal{O}(\alpha^4)\right\} ,$$

$$(7.9)\quad A_2^{(1)}(\theta) = \frac{-4a\sin\theta\sin\gamma}{3\pi}\,\alpha^2\left\{1 - \left[1 - \frac{1}{10}\left(\tfrac{1}{2}+\cos^2\gamma+\cos^2\theta\right)\right]\alpha + \mathcal{O}(\alpha^2)\right\} ,$$

$$(7.10)\quad A_2^{(2)}(\theta) = \frac{4a\sin^2\theta\sin^2\gamma}{45\pi}\,\alpha^4\left\{1 + \mathcal{O}(\alpha)\right\}$$

With the aid of equation (6.9) and these results, the transmission coefficients are found to be

$$(7.11)\quad t_2^{(o)} = \frac{8}{\pi^2\cos\gamma}\left[1 + \frac{\pi^2-36+3\pi^2\cos^2\gamma}{9\pi^2}\,\alpha^2 + \mathcal{O}(\alpha^3)\right] ,$$

$$(7.12)\quad t_2^{(1)} = \frac{32\sin^2\gamma}{27\pi^2\cos\gamma}\,\alpha^4\left[1 - \left(\frac{19}{25} - \frac{\cos^2\gamma}{5}\right)\alpha^2 + \mathcal{O}(\alpha^3)\right] ,$$

$$(7.13)\quad t_2^{(2)} = \frac{128\sin^4\gamma}{30{,}375\pi^2\cos\gamma}\,\alpha^8\left[1 + \mathcal{O}(\alpha^3)\right] .$$

Applying the method of Appendix I, we find, for any integer $m \geqq 0$, that

$$(7.14)\quad f_2^{(m)}(t) = \frac{(-i)^m \epsilon_m \sin^m\gamma\, a^{m+1}(1-t^2)^m}{(2m)!\,\pi}\, \alpha^m \left[1 + \mathcal{O}(\alpha^{m+1})\right]$$

From this expression it follows, as above, that

$$(7.15)\quad \left.\frac{\partial V_2^{(m)}}{\partial z}\right|_{z=o} = \frac{-\epsilon_m m!\, 2^{m+1}(i\sin\gamma)^m}{(2m)!\,\pi a\epsilon} \left(\frac{r}{a}\right)^m \alpha^m \left[1 + \mathcal{O}(\alpha)\right] ,$$

$$(7.16)\quad A_2^{(m)}(\theta) = \frac{4^{m+1} m!\,(m+1)!\,\epsilon_m a(\sin\gamma\sin\theta)^m}{(2m)!\,(2m+2)!\,\pi}\, \alpha^{2m}\left[1 + \mathcal{O}(\alpha)\right] ,$$

and

$$(7.17)\quad t_2^{(m)}(\theta) = (2m)!\left[\frac{4^{m+1} m!\,(m+1)!}{(2m)!\,(2m+2)!}\right]^3 \frac{\epsilon_m \sin^{2m}\gamma}{\pi^2\cos\gamma}\, \alpha^{4m}\left[1 + \mathcal{O}(\alpha)\right]$$

In the special case of normal incidence ($\gamma = 0$) our formulas for $\partial V_2^{(i)}/\partial z$, $A_2^{(o)}(\theta)$ and $t_2^{(o)}$ are in agreement with those given in reference [2] (see* p. 71). We have not succeeded in locating the corresponding expressions for $\gamma \neq 0$. Our results for the higher harmonics seem to be new.

---

[*] A misprint occurs in Bouwkamp's expression for $A_2(\theta)$ which here is denoted by $A_2^{(o)}(\theta)$. It may be rectified by replacing $A_2(\theta)$ by $A_2(\theta)/4$.

APPENDIX I

We shall show that

$$(I.1)\qquad f_1^{(m)}(t) = \frac{\epsilon_m(-1)^m a^{m+1}\cos\gamma\sin^m\gamma\, t(1-t^2)^m}{(2m+1)!\pi}\,\alpha^{m+1}\Big[1 + \mathcal{O}(\alpha)\Big] ,$$

for all integers $m \geqq 0$ when [see equation (2.13)]

$$(I.2)\qquad u_1^{(m)}(r) = i^{m+1}k\cos\gamma\,\epsilon_m J_m(kr\sin\gamma) \Big/ r^m .$$

We begin with the observation that for any[*] $\Delta > 0$ and for some sufficiently small $\alpha_o^{(m)} > 0$, $f_1^{(m)}(t)$ may be expressed as a power series in α,

$$(I.3)\qquad f_1^{(m)}(t) = \sum_{\mu=0}^{\infty} A_\mu(t)\alpha^\mu ,$$

which converges uniformly in t, $0 \leqq t \leqq 1 + (\Delta/a)$, whenever $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$ [see pp. 33 and 34]. As explained on page 34, the A_μ's are polynomials in t.

In the following, it will be necessary to have an estimate of the degrees of $A_o(t)$, $A_1(t),\ldots,A_m(t)$. This is most easily accomplished by expanding the right member of equation (5.4) in powers of α; it is then easy to verify that the degree of $A_\mu(t)$, $\mu = 0,1,2,\ldots,m$ is at most $m + 1$. But we know that $f_1^{(m)}(t)$ is an odd, regular function of t having zeros of order m at $t = \pm 1$ whenever $0 \leqq |\alpha| \leqq \alpha_o^{(m)}$. Accordingly, we may write

---

[*] Observe that the regularity of $J_m(w)$ for all w implies that the inhomogeneous term is regular for all α and t.

$$\text{(I.4)} \qquad f_1^{(m)}(t) = t(1-t^2)^m \sum_{\mu=0}^{\infty} B_\mu(t)\alpha^\mu \; ; \qquad 0 \leqq t \leqq 1 + \Delta/a \; ,$$

where the $B_\mu(t)$'s are constants (independent of t) when $\mu = 0,1,2,\ldots,m-1$ and are even polynomials in t when $\mu \geqq m$.

Now, combining equation (4.3) and (I.2) and introducing the new variables $\eta = (r/a)^2$ and $\xi = t^2$, we find that

$$\text{(I.5)} \qquad \frac{d^{m+1}}{d\eta^{m+1}} \left[\int_0^\eta \frac{\cos\left[\alpha\sqrt{\eta-\xi}\right]}{\sqrt{\eta-\xi}} f_1^{(m)}(\sqrt{\xi}\,)d\xi - \int_0^1 \frac{\sinh\left[\alpha\sqrt{\xi-\eta}\right]}{\sqrt{\xi-\eta}} f_1^{(m)}(\sqrt{\xi}\,)d\xi \right]$$

$$= \frac{i^m k a^{m+2} \cos\gamma\, \epsilon_m J_m(\alpha\sqrt{\eta}\sin\gamma)}{2^{m+1}(\sqrt{\eta}\,)^m} \; .$$

From the form of the left member of this equation and the fact that $J_m(\alpha\sqrt{\eta}\sin\gamma) = \mathcal{O}(\alpha^m)$, it follows immediately that $f_1^{(m)}(t) = \mathcal{O}(\alpha^m)$ when α is small. Consequently,

$$\text{(I.6)} \qquad B_\mu = 0 \quad , \quad \mu = 0,1,\ldots,m-1$$

in equation (I.4). Moreover, comparing coefficients of α^m in both members of this equation, we find that

$$\text{(I.7)} \qquad B_m \frac{d^{m+1}}{d\eta^{m+1}} \left(\eta \int_0^1 \frac{\zeta^{\frac{1}{2}}(1-\eta\zeta)^m d\zeta}{\sqrt{1-\zeta}} \right) = \frac{ka^{m+2}\cos\gamma\,\epsilon_m(i\alpha\sin\gamma)^m}{2^{2m+1}\,m!}$$

from which it follows that

$$\text{(I.8)} \qquad (-1)^m B_m (m+1)! \int_0^1 \frac{\zeta^{m+\frac{1}{2}}}{\sqrt{1-\zeta}}\, d\zeta = \frac{ka^{m+2}\cos\gamma\,\epsilon_m(i\alpha\sin\gamma)^m}{2^{2m+1}\,m!}$$

and hence that

$$\text{(I.9)} \qquad B_m = \frac{ka^{m+2}\cos\gamma\,\epsilon_m(-i\alpha\sin\gamma)^m}{(2m+1)!\,\pi} .$$

Combining this result with equations (I.6) and (I.4), we obtain the desired result, namely (I.5).

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